MU120 Unit 12

The Open University

Mathematics and Computing
A first level multidisciplinary course

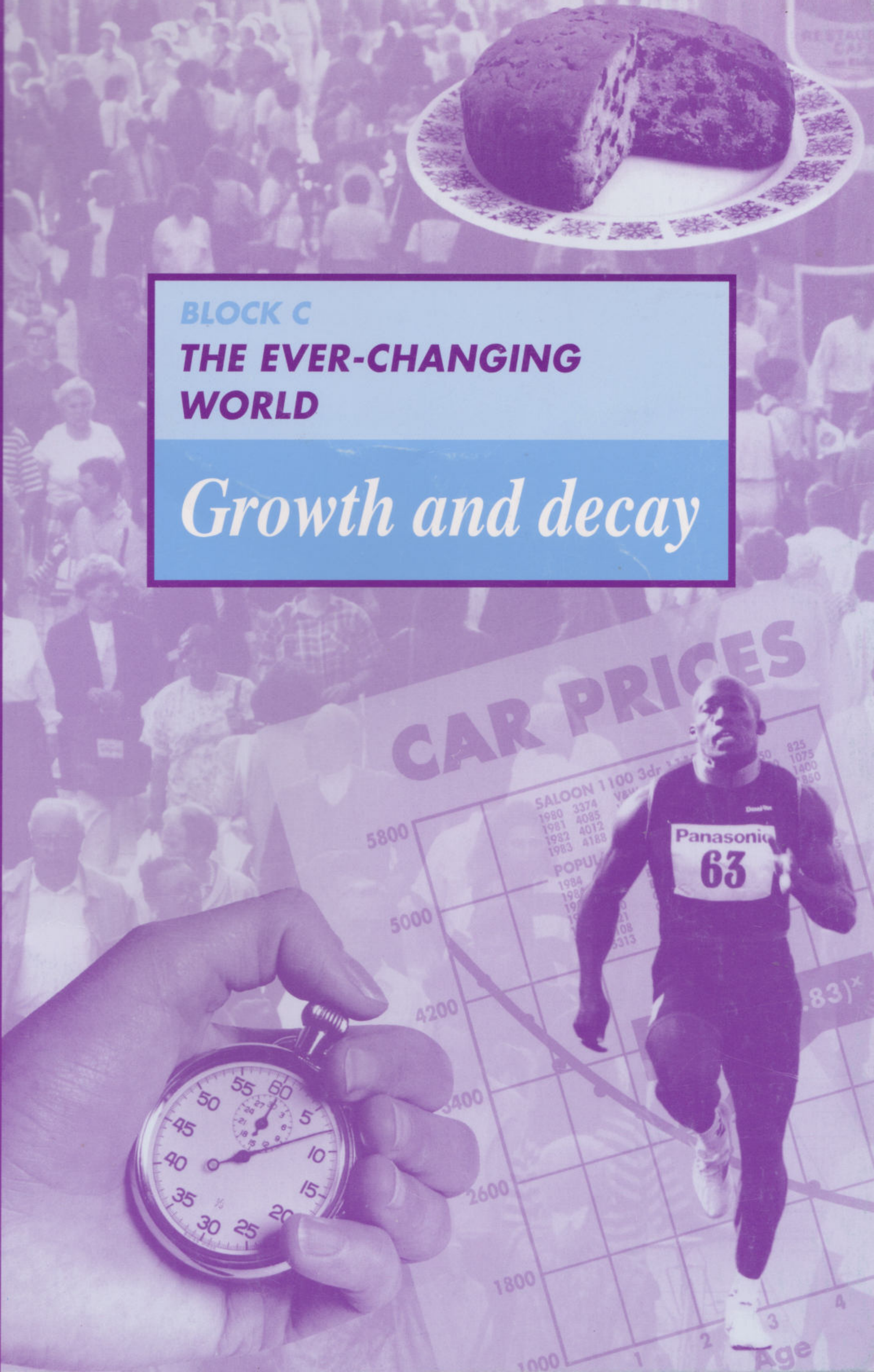
Open Mathematics

UNIT 12

BLOCK C

THE EVER-CHANGING WORLD

Growth and decay



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Prepared by the course team

UNIT 12

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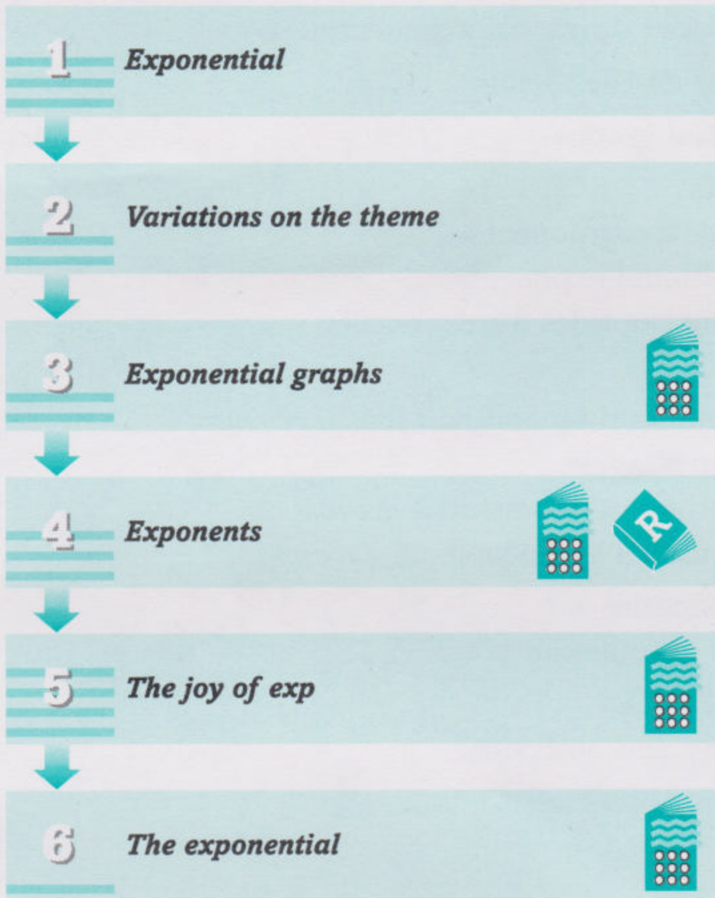
1.4

Contents

Study guide

There are six sections in this unit, of rather uneven lengths. The study plan below shows details of approximate timings. None of the sections make use of audio or video bands, but the calculator is required at many points in the unit. In particular, Section 3, and part of Sections 5 and 6 consist almost entirely of activities based on the calculator. There are some readings about logarithms referred to in Section 4.



Summary of sections and other course components needed for *Unit 12*

Introduction

Back at the beginning of the course, you carried out some investigations using your calculator on the use of scientific notation. One of these investigations involved a rather unlikely sounding but extremely generous monarch, Queen Calcula of Sumwhere, who for some reason best known to herself was prepared to offer you one gold piece on the first day of the month, two on the second, four on the third, eight on the fourth, and so on—ending this generosity on the last day of the month. In the course of the investigation, you had to find out how many gold pieces were on offer each day. It turned out that on the last day of the month (assuming it to be 31 days long) there were 1 073 741 824, or about 1.07×10^9, gold pieces going. In another investigation, you were asked to calculate how many ancestors you have, going back to the 30th generation.

Both of these investigations involve some number (of gold pieces, of ancestors) which grows step by step by a simple multiplicative law: the number of gold pieces offered by Queen Calcula on any day of the month is *twice* the number she offered on the previous day; the number of your great-great-. . . -great grandparents is *twice* the number of your one-great-fewer great-great-. . . -great grandparents.

These are examples of change according to a pattern which is common and important. It is called *exponential* growth; why it is given this name will be explained later. The term 'exponential' applies to any process in which something changes according to the rule that the quantity there is of it at any stage is a fixed multiple of the quantity of it at the previous stage. It so happens that in both of our examples the factor by which the quantity grows at each stage (the multiplier) is 2. This is a coincidence: the word 'exponential' is used whatever the factor may be. In *Unit 9*, you saw that the frequency of a note sounding one semitone higher than the previous note in the Western chromatic scale is always $\sqrt[12]{2}$ times the previous frequency.

Exponential change can result in decrease, or decay, rather than growth, if the quantity involved decreases step by step rather than increasing—provided it decreases by a fixed factor (that is to say, the factor by which the quantity gets multiplied must be less than 1). Again, from *Unit 9*, you saw that to produce a note a semitone higher, the length of string must be decreased by a multiplicative factor of

$$\sqrt[12]{\frac{1}{2}} = \frac{1}{\sqrt[12]{2}}$$

which is a number less than 1. And the string length gets shorter.

The purpose of this unit is to describe and explain the special features of exponential change, and give examples, both serious and more light-hearted, of situations which lead to exponential change. The use of an exponential relationship (repeated multiplication by a fixed factor) to

describe a situation leads to a kind of model which is called an exponential model. You had a brief encounter with an exponential model in the previous unit. Several of the important properties of exponential change have occurred here and there earlier in the course, though you may not have recognized them as such. So this unit will call on knowledge and experience you have already gained, as well as introducing new ideas.

When an idea or technique is first encountered, it tends to be fuzzy, indistinct and imprecise. Gradually, as further experience is gained, it seems to take shape, until it reaches a reasonably stable and usable form. It is often useful to say out loud what something is about.

In speaking out loud, you may notice yourself making connections, associating ideas, giving examples, stating properties, and so on. The very fact that that you are able to make so many connections could indicate to you that the idea is 'crystallizing'. It is possible that, at some later time, you may be forced to reconsider your ideas as new and different views are brought to your attention; but the important aspect is that you are remaking connections and so making the ideas your own.

What can you do that can help in transforming ideas you meet in the course into your own? If you think about how you approach a new idea, the first stage that is important seems to be that, while the idea is fuzzy, you see what it is about. This is long before you can talk coherently about it to someone else; and even after you can talk about it, it is still very hard to write down what you understand in a coherent fashion. As you work through this unit connecting new ideas, think about these transitions from seeing something, to saying what you see, to recording it.

Activity 1 *Seeing, saying and recording*

Whenever you encounter an idea, try re-stating it in your own words. At first, you may find it hard to use words other than the ones in the text—but try to use particular examples and then tell yourself in which way the particular examples are special cases of the more general idea. When you have had several goes at saying it, try to capture it as succinctly as possible, using symbols, pictures or words. Use the Learning File sheet to record you ideas and refer back to it every so often as you need to, and, where you want to include terms in your Handbook, use your definitions as entries.

Activity 2 *Handbook work*

There are two Handbook-type sheets associated with this unit: one is for recording definitions/explanations of new terms and the other for describing techniques. As you work through the unit, make appropriate entries.

1 Exponential growth

Aims This section aims to introduce a number of examples of exponential growth (and decay), and show how to describe this kind of change mathematically. ◇

1.1 Examples of exponential growth

Here are some further examples of situations involving change, all of which show the same pattern of repeated multiplication by a constant factor which is the characteristic of exponential growth or decay.

Chain letters

A chain letter asks (or threatening ones demand) its recipients to send a copy of the same letter on to a certain number of people: five people, perhaps. These new recipients are each required to send further copies on to five of their friends (or enemies). So in each 'generation' of such a chain letter, five times as many people will receive a letter as received one in the previous generation. The number of letters in each generation grows exponentially, with a multiplicative factor of five, so long as all of the recipients obey the rules.

Greenfly

Why do roses get covered so quickly with greenfly? Because they breed so fast! Greenfly can reproduce asexually, and do so at the rate of roughly one offspring per hour. Each individual in each new generation is able to reproduce when it is just twenty-four hours old; each individual produces (roughly) the same number of young. The population of greenfly on a rose bush grows exponentially—until the gardener, or the ladybirds, do something about it.

Trees

In the autumn, a certain number of buds develop at and near the end of each twig of a deciduous tree. Next spring (provided it is not destroyed by frost or eaten by bullfinches), each of these buds sprouts into a new twig. One obvious question is: how does the number of buds a tree has change with its age? This is one of the elements which governs the shape of the tree, the geometry of the new twigs (their lengths and the angles between them) being the other. The growth of an idealized tree—one not subject to the vagaries of the weather—according to a law of growth like the one just described can easily be illustrated by using a computer. Some stages in the growth of a computer-generated tree are shown in Figure 1. The number of buds on such an idealized 'tree' grows exponentially.

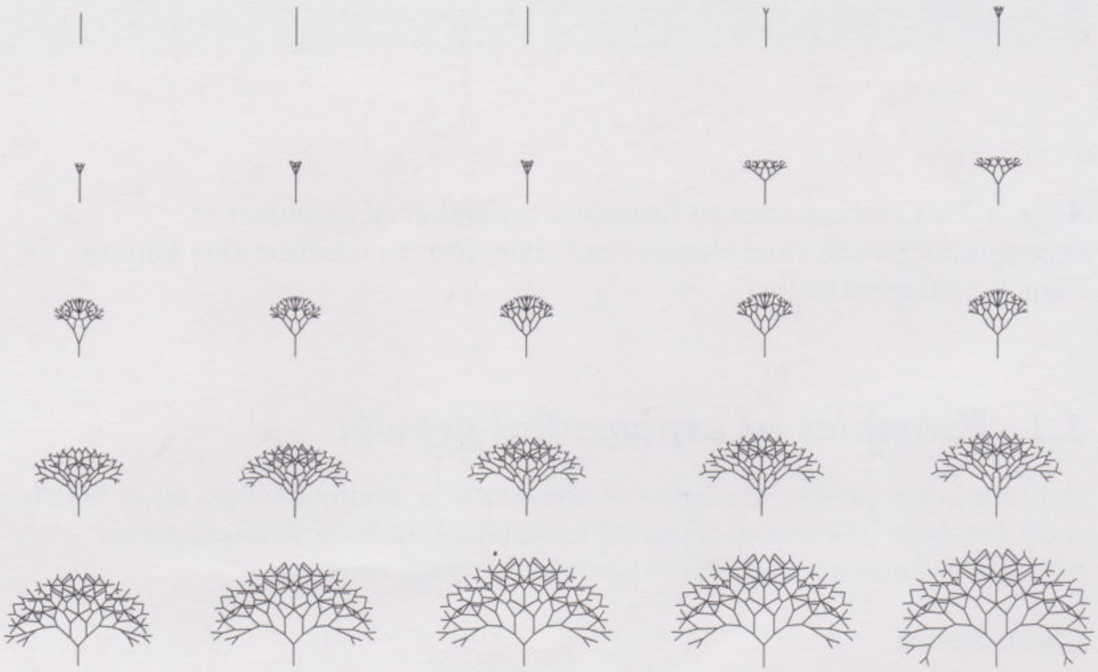


Figure 1 Computer-generated trees

Interest

A sum of money in a savings account which attracts compound interest grows exponentially.

Radioactive decay

Some atoms occur in a number of different forms, called *isotopes*. The difference lies in the composition of the atom's nucleus. The nucleus of an isotope may contain too many sub-particles (such as neutrons) to cohere together satisfactorily; the isotope is then said to be unstable, and may spontaneously convert itself into something else, releasing some of the offending particles and some energy in the form of α, β or γ rays. An isotope which behaves in this way is said to be *radioactive*, and the process by which radioactive isotopes break down is called *radioactive decay*. Imagine a collection of atoms of a radioactive isotope of some element. The number of radioactive atoms in it will decrease over time as more and more of them disintegrate. It has been found, both theoretically and by empirical observation, that radioactive isotopes decay exponentially.

When physicists first began investigating the structure of the nucleus, they discovered one new ray or particle after another, and named each with a Greek letter. It used to be said that in order to be a nuclear physicist all you needed was a detailed knowledge of the Greek alphabet. Such knowledge is quite useful for mathematics, too.

The Sierpinski carpet

Begin with a black square; subdivide it into nine smaller squares and paint the middle one white; subdivide each of the remaining eight black small squares into nine yet smaller squares and paint the middle one white; repeat *ad nauseam* (Figure 2). As the process is repeated, the area of the black part of the carpet decreases exponentially.

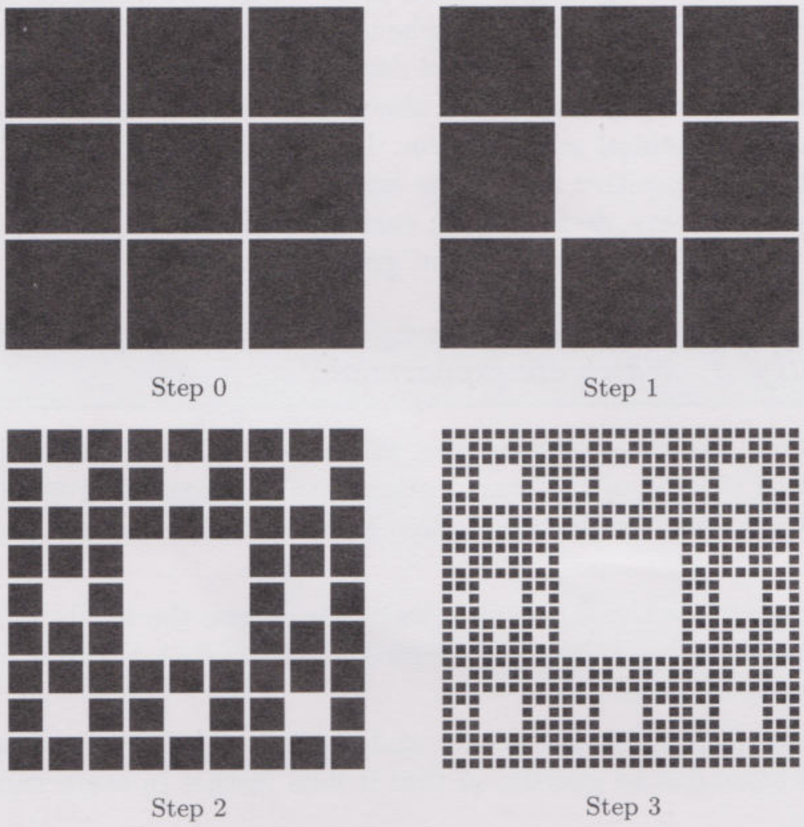


Figure 2 The Sierpinski carpet

These examples all involve the idea that in going from one stage to the next a quantity gets multiplied by a fixed factor. Unfortunately, it seems to be quite difficult to find words to describe the process under investigation which will be appropriate in all the many different situations in which it occurs. 'Quantity' and 'stage' are awkward and bland. One common situation in which exponential growth is important is exemplified by the greenfly. Words like 'population' and 'generation', which are obvious ones to use when talking about numbers of living organisms, can with a certain amount of poetic licence be used in all sorts of other situations too.

So the words 'population' and 'generation' will be used metaphorically to cover all kinds of situation which will arise in this unit. Thus, when discussing exponential change you will be dealing with a collection of things of a particular kind, which will be called a *population*. You must bear it in mind that this so-called population might consist of francs, sticky-buds, postcards or atoms, as well as people, microbes or greenfly. The number of these things, which will be called the *size of the population*, is supposed to be changing (mostly, but not necessarily, over time) in a discrete, step-by-step manner; each step will be called a *generation*. The defining characteristic feature of *exponential* change is that the size of the population in one generation is a fixed multiple of the size of the population in the previous generation.

The unit often uses the general term exponential *growth*, despite the fact that exponential change may be decay rather than growth, just to save

repeatedly having to write 'growth or decay'. The word 'change' will not really be enough by itself, because when something is changing it can increase for one part of the time and decrease for another part of the time. The factor by which the population size gets multiplied in every generation is called the exponential *growth factor*. If the exponential 'growth' factor, which is always a positive number, is actually a number less than 1, then the population decays, decreasing at each stage; this is an example of using the word 'growth' as a shorthand for 'growth or decay'.

Activity 3 Which are exponentials?

Here are some examples of situations which involve change, from earlier units. Which (if any) of them are examples of exponential growth? For those that are, identify the population, the generation, and the growth factor.

(a) A culture of bacterial cells in a test tube, where the height of the top of the cell culture changes from 1 mm to 2 mm to 4 mm to 8 mm in successive minutes.

(b) An object falling from a height, and accelerating downwards with the acceleration due to gravity, so that it falls further in every successive second.

(c) 'As I was going to St Ives,
I met a man with seven wives;
each wife had seven sacks,
each sack had seven cats,
each cat had seven kits;
kits, cats, sacks, wives:
how many were going to St Ives?'

The change in question is in the number of things *coming from* St Ives when moving from line to line in the poem.

(d) The world record time for running the mile, which decreased from about 4.2 minutes in 1920, to 4 minutes in 1950, to 3.8 minutes in 1980.

Activity 4 Find some yourself

Think of some examples of exponential growth of your own. If you keep an eye on the newspapers and TV, you will probably find some. In fact, the media are quite fond of the word 'exponential': it is not unusual to come across statements like: 'the amount of computing power available per thousand pounds is growing exponentially'. One thing you might consider is whether the word is used correctly in situations like this. On the other hand, true examples of exponential growth are not necessarily identified only by the use of the word.

1.2 Formulas for exponential growth

How can you calculate how many ancestors you have had in each previous generation? Well, in the first generation back there are 2 (your biological parents); in the second, there are 4 (your grandparents, or your parents' parents); in the third, there are 8 (your great-grandparents, or your grandparents' parents); and so on. So the pattern goes: 2; $2 \times 2 = 2^2 = 4$; $2 \times 2^2 = 2^3 = 8$; $2 \times 2^3 = 2^4 = 16$; and so on. This can be expressed in a way which avoids the irritating 'and so on', by using a general formula and a variable which stands for the generation being described. The number of ancestors in the nth previous generation is 2^n. Substitute in the first few values of n to check the formula does in fact give the above particular values.

Activity 5 Chain letters

Consider a chain letter which involves the instruction 'send a copy of this letter on to five of your friends'. The person who sets the chain letter in motion sends out the first five letters; the recipients then each send out five letters, and so on. How many letters are sent at the 3rd stage? At the 17th stage? At the nth stage? In other words, if this is regarded as a population of letters, what is the size of the population in the nth generation?

In order to write the results of these examples as formulas, some suitable notation will be required. The symbol n has already been introduced to stand for the number of the generation (so that, for example, $n = 3$ corresponds to the third generation). A symbol for the size of the population will also be needed; what could be more natural than to call it P. So for the problem of the number of ancestors, write:

$$P = 2^n$$

to mean that the size of the population (the number of ancestors) in previous generation number n is 2^n. So if, for example, you want to calculate the number of ancestors you had in the tenth generation back, then you just have to substitute 10 for n in the first formula, to get $P = 2^{10} = 1024$. In a similar fashion, the formula for the chain letter problem is:

$$P = 5^n$$

P is now used to represent the number of letters sent out at the nth stage (generation).

Activity 6 Chain mail

How many chain letters get sent in the 10th generation?

The size of the population depends on which generation you look at, or in other (mathematical) words the size of the population is a function of the generation. When it is desirable to emphasize this dependence, as it sometimes is, the notation is extended a little: write $P(n)$, rather than just P, for the population size. This can be read as 'P of n', or as 'the population size in the nth generation'. This extended notation brings many benefits, some of which follow from the fact that it says explicitly which generation is being talked about. You can take advantage of this to create a useful shorthand for a phrase like 'the number of ancestors you had in the tenth generation back': this is simply $P(10)$ (where $P(n)$ gives the size of the population of ancestors). And the formula for the population size can be written as $P(n) = 2^n$.

It is now possible to explain the origin of the term 'exponential' to describe this pattern of growth. It comes from the use of the word 'exponent' to refer to the variable n in an expression like 2^n or 5^n. The word 'exponential' simply means 'involving an exponent'. When you raise the number 10 (say) to some power, the power to which you raise it is called the exponent; and so a function in which the dependent variable appears as an exponent is called an exponential function.

According to the dictionary, the word 'exponent' comes from the Latin verb *pono*, meaning 'I place', with the prefix *ex*-, meaning 'out'. It is an exponent because you place it out (of the way).

You have now met several examples of exponential growth, so you should be beginning to get a clearer idea of what it is. When learning about a new concept, it often helps to have to hand an example of what it is not as well as examples of what it is. Here is a situation which leads directly to a comparison of non-exponential and exponential growth. The topic is a financial one: interest earned by a savings account, or similar investment.

Example 1 *Savings account: simple interest*

When money is saved in an account which earns *simple interest*, the interest earned is regarded separately from the deposit, and does not itself gain interest. Consider a savings account which offers a fixed annual (currently unrealistic) simple interest rate of 10%; the interest is credited to the account at the end of every year since the sum was first invested. Suppose that a sum of £200 is deposited initially. How does the amount of money in the account grow?

At the end of one year, the interest is:

$$£200 \times \tfrac{10}{100} = £200 \times 0.1 = £20$$

and the amount of money in the account is now £220. At the end of the second year an *identical* amount of interest is earned, so the balance of the account increases to $£200 + 2 \times £20 = £240$. This is repeated at each year-end: if $P(n)$ is the balance of the account at the end of the nth year, then a formula for computing $P(n)$ is:

$$P(n) = 200 + 20n$$

As you can see, this is not exponential growth; it is in fact an example of linear growth, as you should recognize as a result of your study of *Unit 10*.

Here is a contrasting case.

Example 2 Savings account: compound interest

Most savings accounts pay *compound interest.* This means that once the interest is earned, it is added to the sum already in the account, and then this new *total* sum earns interest. Consider the case discussed in the previous example, but suppose now that the account pays compound rather than simple interest. As before, the interest is added to the account annually. How does the balance grow this time?

At the end of the first year, the interest is £20 as before, giving a total of £220 in the account. However, the interest at the end of the second year is calculated on this new total sum, and so is:

$$£220 \times 0.1 = £22$$

The total amount in the account at the end of the two-year period is therefore:

$$£200 + £20 + £22 = £242$$

You should be able to see that you will always do better if interest is calculated on a compound rather than simple basis—assuming the interest rate paid on the two types of account is the same.

Showing how to calculate the balance in the account at the end of each year in a slightly different way will enable you to see that compound interest is an example of exponential growth. Suppose that the amount of money in the account at the end of one year is $£P$. Then, at the end of the next year, the amount of money will have grown to $£(P + 0.1P)$, of which $£P$ is the balance from last year carried forward, and $£0.1P$ is the interest earned during the year. Now $P + 0.1P = 1.1P$, so the amount of money grows each year by the multiplicative factor 1.1. So the amount of money grows exponentially.

In one respect, this is not quite the same as the two examples of exponential growth that have been discussed earlier in this subsection. If you think about how the balance grows, you will see that after 1 year it is 200×1.1, after two years it is $200 \times (1.1)^2$, after three years it is $200 \times (1.1)^3$, and so on. So the amount of money (in £) in the account at the end of the nth year, $P(n)$, is given by:

$$P(n) = 200 \times (1.1)^n$$

You could write this as just $200(1.1)^n$—there is no real need for the $\times$. However, to leave it out may lead to misunderstandings, so leave it in for the time being, until you are confident that no confusion will arise.

This differs from the examples you have seen already by the presence of the coefficient 200, which is the sum initially invested. (You might find it helpful to think of the term $(1.1)^n$ as representing the amount to which each £1 initially invested has grown; then the £200 will grow to 200 times this amount.)

So if you are at all interested in money, and want a way of remembering what is special about exponential growth, you could say to yourself 'it is compound interest that gives exponential growth; simple interest gives only linear growth'.

Activity 7 *Interest—simple vs. compound*

(a) Simple interest is added annually to a £500 deposit, and at the end of five years the interest gained is £108.50. What is the annual interest rate?

(b) Suppose instead that interest is compounded annually at a fixed rate. Use your calculator to estimate the rate, assuming the interest received over the five years is still £108.50.

Activity 8 *More on money*

You put £p in a savings account which gives 10% compound interest annually. How much is there in the account at the end of the nth year after you first deposit the money?

Activity 9 *It's the rich ...*

The chief executive of a recently privatized company wishes to give her newly born child a gift of £1 000 000 when he comes of age in eighteen years time. She finds an account which guarantees to pay 5% per year for the whole of this period, compounded annually. How much must she invest now in order to achieve her target?

Activity 10 *Simplified greenfly*

(a) The simplified greenfly is a mythical insect—but designed to help you understand how real populations (of insects, birds, yeast cells, crocodiles and possibly even people) grow. The simplified greenfly breeds asexually. Each adult (simplified) greenfly has three offspring and then dies. The offspring reach maturity on the day after their birth, and each has three offspring, and then itself dies that day; and so on. On 1 May, I find that there are seventeen newborn simplified greenfly on my favourite rose bush ('Boule de Neige', since you ask). Work out a formula for the number of simplified greenfly there will be on the bush at any later date during the summer.

(b) Actually those were oversimplified greenfly: according to a more realistic model, each simplified greenfly has twelve offspring, but only a quarter of them survive to maturity (the rest are eaten by ladybirds). What difference, if any, does this make to the formula for the number of simplified greenfly?

Here is another situation which leads to exponential growth, in the slightly more general version which arises with compound interest and simplified greenfly.

Example 3 Mathematical snowflakes

There is a rather beautiful geometrical object called a snowflake curve. A snowflake curve is constructed in a step-by-step fashion as follows. You start with an equilateral triangle (a triangle whose sides are all the same length). You divide each side into three equal parts, draw another equilateral triangle (each of whose sides will be one third of the length of the original one) on the middle section, and then erase the middle third itself. The result is now a six-pointed star. You then take each side (that is, each straight line section) of the star, divide it into three equal parts, and replace the middle third as before. You now have a shape which has no name, but is beginning to look like a snowflake (or, to be more accurate, like what a snow crystal looks like under a microscope).

But now you repeat the whole process again ... and again ... and so on, as many times as you have patience for, and as long as your pencil is fine enough. (Strictly speaking, it is the ultimate curve that you get—or you have to imagine getting, if you keep on repeating the process of subdividing the sides indefinitely often—that is called a snowflake curve.) It may perhaps seem a bit odd to call the figure that you get at each stage in the construction a 'curve', since it is made up of straight lines, but the resulting shape is more complex than any of the stages along the way.

Figure 3 shows the first few stages in the construction of a snowflake curve.

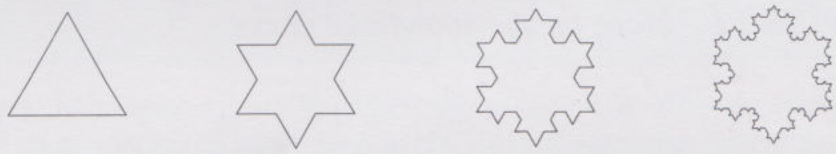


Figure 3 Constructing a snowflake curve

The snowflake curve is 'self-similar': if you take any little bit of it, and magnify it suitably, what you get is essentially indistinguishable from a large bit of it. The snowflake curve is like a cauliflower in this respect. It is also like the leaves of some trees (to take a two-dimensional rather than three-dimensional botanical example): some leaves have toothed edges, and on close inspection the teeth themselves are toothed, and so on. You could imagine adapting the method of construction of the snowflake curve so as to obtain pretty good likenesses of such leaves as hazel and lime.

This idea was mentioned in the audio mathematical diary in *Unit 1*.

Leaves grow at their margins; that is to say, new cells grow only from cells on the edge of the leaf. It is possible to imagine a mechanism for controlling growth which encourages new cells to grow in certain stretches of the margin and inhibits them from growing in others. But this is just how the region inside the snowflake curve grows: 'cells' in the middle third of any straight-line segment of the edge are encouraged to grow; the rest are prevented from growing.

The 'population/generation' terminology is a bit forced here, but useful nevertheless. The population is the collection of lines that go to make up the figure at any generation of the construction; the population size, here, is the total length of all those lines.

One interesting question about snowflake curves is how long they are: how does the perimeter of a snowflake curve—the total length of its sides—grow, generation by generation?

At each stage of the construction, some straight lines are replaced with another set of lines with kinks in them. The general replacement step is shown in Figure 4.

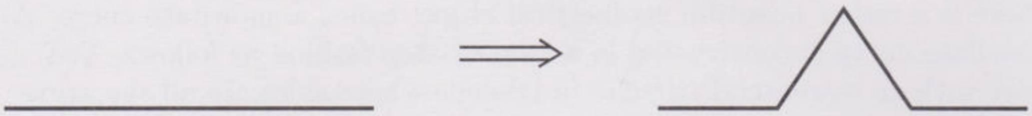


Figure 4 Constructing a snowflake curve: replacing a line

For each straight line, the middle third is replaced by something twice as long; so the three-thirds length of the original line turn into four thirds as a result of this replacement. Each line in the figure is replaced by something $\frac{4}{3}$ times as long. The total length of the figure is therefore multiplied by the factor $\frac{4}{3}$ at each generation. Suppose, for example, that the original equilateral triangle has sides of unit length, so that its perimeter is 3; then the lengths of successive figures in the construction of the snowflake curve are:

$$3 \times \left(\tfrac{4}{3}\right); \qquad 3 \times \left(\tfrac{4}{3}\right)^2; \qquad 3 \times \left(\tfrac{4}{3}\right)^3; \qquad 3 \times \left(\tfrac{4}{3}\right)^4; \qquad \ldots$$

The length grows exponentially, with growth factor $\frac{4}{3}$; there is an overall coefficient of 3 in the formula for the length, which is simply the perimeter of the original triangle.

Activity 11 *More on the snowflake curve*

(a) This time, specify $P(n)$ to be the length of the perimeter of the figure created in the nth generation. Write down an explicit formula for $P(n)$.

(b) Suppose that the sides of the original triangle were each $\frac{1}{3}$ of a unit long. What would the length of the perimeter of the nth generation figure be in this case? Suppose the *perimeter* of the original triangle is 5 units long—what would the length be then?

(c) It should be becoming clear to you that it would be relatively straightforward to write down a formula which gives the perimeter of the figure obtained in the nth generation in terms of both n and the perimeter of the original triangle. Write l for the perimeter of the original triangle. Convince yourself that:

$$P(n) = l \times \left(\tfrac{4}{3}\right)^n$$

The formula for the amount of money in the savings account which started off with a deposit of £200 and earned compound interest at 10% per annum is:

$$P(n) = £200 \times (1.1)^n$$

This formula allows you to calculate the balance at the end of the nth year, where $n = 1, 2, 3, \ldots$. But there is a little more to be got out of the formula, by using the fact that $(1.1)^0 = 1$. (The reason for this will be explored later in the unit.) So, putting $n = 0$ gives $P(0) = 200$. That is to say, $P(0)$ is the amount of money originally deposited. Likewise in the case of the snowflake curve, $P(0)$ is the length of the perimeter of the triangle with which the construction starts.

The characteristic feature of exponential growth is the fact that the size of the population in one generation is a fixed multiple of the size of the population in the previous generation, the factor by which the population size gets multiplied in every generation being called the *growth factor*. Consider how to express this characterization in mathematical language—that is, in symbols. Call the growth factor b. The population size in the nth generation is denoted by $P(n)$. The population size in the next generation is $P(n+1)$ (the next generation after the nth is the $(n+1)$th). The population size in any one generation is b times the population size in the previous one:

$$P(n+1) = b \times P(n)$$

This formula expresses the characteristic property of exponential growth.

The discussion in the previous two paragraphs leads to the following conclusion. Suppose you are dealing with an exponentially growing population with growth factor b. Then the population size $P(n)$ obeys the law $P(n+1) = b \times P(n)$. But it is also the case, as the examples show, that $P(n)$ is given explicitly by the formula $P(n) = a \times b^n$, where $a = P(0)$ is the initial size of the population (its size at the start). So you have seen how to express in mathematical language both the rule which governs exponential growth, and the explicit formula which enables you to calculate the actual size of a population which is growing exponentially, *provided* you know both the constant growth factor and the initial size of the population.

1.3 What does it all mean?

No doubt by now you have got into the habit of taking time at the end of every section of a unit to reflect on what you have just learnt or discovered, to fix it in your mind, to think about its significance, and to consider how it fits in with what you know already. There are one or two important things of a general nature that need saying, by way of summary and clarification.

In this section, you have been concerned with a quantity called P, or often (for emphasis and clarity) $P(n)$. What does this (and any other similar symbolization) stand for?

▶ What *is* $P(n)$?

(Resist the temptation to read straight on. Just how *would* you answer this question? One good way of approaching it is to decide what you

would say if a fellow student asked you the same question, in genuine puzzlement: 'Oh dear, [insert your own name here] give us a hand, I can't make head or tail of this: just what *is* this thing called $P(n)$?')

There are at least *three* ways of answering the question 'what is $P(n)$?', each of them valid, each of them useful, each of them appropriate to a different set of circumstances. Here they are:

(a) $P(n)$ is the size of the population in the nth generation;

(b) $P(n)$ is the exponential expression ab^n;

(c) $P(n)$ is the quantity determined by the relation $P(n+1) = bP(n)$.

You should have got used to the various formulas by now, so we shall now revert to conventional practice and leave out the × signs.

The first of these would be replaced by something rather more specific in any particular example—for example, in a financial problem you might have '$P(n)$ is the amount of money in the nth year in a savings account which earns interest which is compounded annually'. Likewise, the numerical values of a and b in the second and third answers will depend on the specific case. (Strictly speaking, in the third answer it is necessary to add that the initial value of P, namely $P(0)$, is a; otherwise $P(n)$ will not be completely *determined*. But this is not important for present purposes.)

The first of these is the answer to the question 'What is $P(n)$?', in the sense 'What does $P(n)$ represent in the context of the problem you are trying to solve?' You could perhaps say 'What does $P(n)$ *mean*?' In the second, the question 'What is $P(n)$?' is interpreted as 'What explicit expression is $P(n)$ shorthand for?' Alternatively, 'What does $P(n)$ *refer to*?' The third answer is the answer to the question 'What is $P(n)$?' in the sense 'What is distinctive about $P(n)$?' or 'How would you *characterize* it?' The final answer is perhaps the least obvious of the three, though on consideration it may be most important; at the least, it is the point which connects the other two.

Mathematics habitually takes advantage of the possibility of interpreting symbols in different ways—of the plurality or ambiguity of symbols, if you like (though 'ambiguity' does not mean 'imprecision', since each of the interpretations is itself precise)—when making connections between different aspects of a problem. For example, any of the following might be said: 'the size of the population in the nth generation is given by an exponential expression', or more naturally 'the population grows exponentially'; or 'the population size $P(n)$ satisfies the relation $P(n+1) = bP(n)$'; or 'the solution to the relation $P(n+1) = bP(n)$ is an exponential'.

Finally, these remarks connect to the description of mathematical modelling in Section 1 of *Unit 10*. You have probably recognized that you have been reading and thinking about the mathematical modelling of a number of similar processes: so similar that, first, they can all be roughly described as being concerned with the growth or decline of a population, and second, they are all modelled by the same kind of mathematics: exponentials.

When setting up a model, the first step is to specify its purpose: in all of the cases under discussion that purpose is to find out how the size of a population depends on the number of generations for which it has been developing. The first of the answers to the question 'What is $P(n)$?' links it with that purpose. One result of sorting out the purpose of the model is that when you have done so, you know what the relevant variables are; the first answer to the question also specifies the variables.

◁ *Specify purpose* ▷

The next step in the modelling process is to create the model; that is to say, after choosing variables you must find a mathematical relation among them, using what you know about the processes at work in the situation you are modelling. For the population model, this leads to the relation $P(n+1) = bP(n)$. Next, you have to do the mathematics. The end result of doing the mathematics in this case is the particular expression for $P(n)$ as an exponential. So the three different answers to the question 'What is $P(n)$?' can be connected to different stages in the creation of a model.

◁ *Create model* ▷

◁ *Do the maths* ▷

Of course, modelling goes further than doing the mathematics. The next stage is to interpret the results. One way of qualitatively interpreting the results of the exponential model of population growth—that is, of getting some idea about its significant features—is to graph the exponential function. This will be tackled in Section 3. But, first, there is a further example of the exponential model to be discussed (in the next subsection), and then some variants of it to be considered (in the following section).

◁ *Interpret results* ▷

1.4 Collared doves: an exponential model

This subsection involves an example of the use of an exponential model to describe the growth of a population of living things—in this case, birds. It is taken from the book *Models in Biology* by D. Brown and P. Rothery (Wiley, 1993). (A few minor changes have been made to bring the text into line with this unit.)

When dealing with processes which occur in nature, modelling becomes a bit more complicated than it is for the kind of situation under controlled conditions in the mathematical laboratory, like those dealt with already. One very useful technique is illustrated in the passage below: the use of what the authors call a 'word model'. They express the fundamental relationship between the number of breeding females in one generation and the number in the previous generation in words, before they attempt to formulate it algebraically. This should help you to understand the derivation of the formula $P(n+1) = (k + \frac{1}{2}ml)\,P(n)$ which is the key relationship expressed in the mathematical model. Remember, back in *Unit 8*, the transition from the word formula for a 'Think of a number' sequence to the algebraic expression for the same thing. It is a good technique to use when you are formulating a mathematical model yourself.

Following the box is an activity which will test your understanding of the rather complex modelling described in it, and should suggest to you kinds of things to look out for when reading similar accounts in future. You may care to look at Activity 12 before reading the boxed text.

Collared doves

Following a dramatic expansion of its range across Europe from the Balkans to the North Sea in under twenty years, the collared dove, *Stretopelia decaocto*, bred for the first time in England in Norfolk in 1955. The subsequent increase in numbers over the next ten years is summarized in Figure 5.

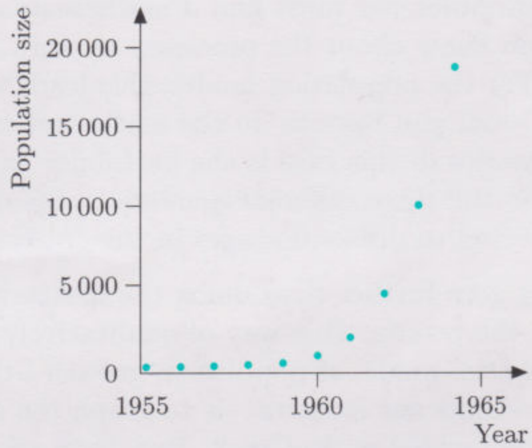


Figure 5 Number of collared doves in the British Isles: the population size is the number of adults at the beginning of the breeding season

By 1964, the species had reached most areas of Britain and the population size, measured as the number of adults at the beginning of the annual breeding season, had increased from four to an estimated 18 855. To explain the rapid increase in population size, a simple model will be built which incorporates information on survival and reproduction. Each year, a number of breeding pairs rear young which are capable of breeding as adults in their first year after fledging (learning to fly). The number of adult females in the breeding population in a particular year is made up of the surviving adults from the previous year plus the number of their offspring which survive to be recruited. A word description for the population change is then:

$$\begin{aligned}&\text{number of adult females in breeding population in year } (n+1)\\ &= (\text{number of adult females in breeding population in year } n\\ &\quad\times \text{proportion of adults surviving to year } (n+1))\\ &\quad+ (\text{number of young females produced in year } n\\ &\quad\times \text{proportion of young surviving to year } (n+1))\end{aligned}$$

Developing this general equation for population change involves considering the particular simple case in which survival rates and birth rates do not vary from year to year. A constant proportion k of adult females breeding in one year survive to breed in the next and a constant proportion l of young birds survive to become adults and breed at the end of their first year. Each female produces a constant number m of young, of which half are assumed to be females.

The above word model can now be written as an equation for the number of adult females $P(n)$ in the breeding population of year n given by:

$$P(n+1) = kP(n) + \tfrac{1}{2}mlP(n)$$

or

$$P(n+1) = \left(k + \tfrac{1}{2}ml\right) P(n)$$

The formulation therefore reduces to the simple model of exponential growth given by:

$$P(n+1) = bP(n)$$

and shows that the exponential growth factor b is related to the birth and survival rates by the relation:

$$b = k + \tfrac{1}{2}ml$$

Hofstetter studied collared doves in Germany in 1954 and estimated average survival rates of 86% for adults and 60% for juveniles in their first year. Collared doves lay clutches of two eggs, although most pairs lay two clutches and some lay more, so that each pair has the potential to rear 4 young. For a population in which $k = 0.86$, $l = 0.60$ and $m = 4$, the calculated exponential growth factor is 2.06 per annum.

Activity 12 *Investigating the model*

(a) The model was developed for the female population, but the data on the graph are for the whole population. How is the size of the whole population related to the size of the female population? What is the exponential growth factor for the whole population?

(b) From the graph, the population size in 1960 is very close to 1000 (before this year the numbers are too small to estimate with any confidence). Compare the model's predicted results with the measured population sizes for the years 1961 to 1964, taking 1000 as the starting value of the model population (that is, the idealized population described by the model, which is supposed to approximate the real population).

(c) Explain why the values of k, l and m for the model population are taken to be 0.86, 0.60 and 4, respectively. How is the value 2.06 for the growth factor of the model population obtained?

(d) The population size is defined to be 'the number of adults at the *beginning* of the breeding season'. Why is it necessary to be so careful about the definition of the population size? How does the population size vary during the year, rather than from one year to the next?

(e) What does the expression $kP(n)$ correspond to in the description of the model in words? Explain how the term 'number of young females produced in year n × proportion of young surviving to year $(n + 1)$' is translated into symbols to complete the algebraic expression of the model.

(f) Can you suggest any reasons why the population of collared doves in Britain and Ireland grew so rapidly once the doves had become established? Do you think that the population could continue to grow exponentially indefinitely into the future?

In summary, this section has discussed exponential growth as a distinctive and important kind of change, frequently met both in purely mathematical problems and in real life. The characteristic property of exponential growth is expressed in the formula which gives the population size $P(n+1)$ in the $(n+1)$th generation in terms of the population size $P(n)$ in the previous generation: it is $P(n+1) = bP(n)$, where the positive constant b is the growth factor. There is an explicit formula for $P(n)$ for a population which grows exponentially: $P(n) = ab^n$, where a is the initial population size.

Outcomes

After studying this section, you should be able to:

- ◇ explain what is meant by 'exponential growth', recognize examples of exponential growth, and distinguish exponential growth from other types of change, such as linear or quadratic change (Activities 3, 4, 7);
- ◇ write down the general formula for the population size of a population growing exponentially (Activities 5, 6, 8, 9, 10, 11);
- ◇ interpret descriptions of exponential models (Activity 12);
- ◇ see, say and record ideas.

2 Variations on the theme

Aims This section aims to help you find out about a type of growth which is different from, but related to, exponential growth, namely the accumulation of the results of exponential growth. ◇

The problem of Queen Calcula's gift was left in an unsatisfactorily incomplete state. Surely, if you were considering an offer like Her Majesty's, what you would want to know is not just the amount of money she would pay you on the last day of the month, but the total amount she would pay you through the month; that is, the sum of all the payments on the different days of the month. How could you work out this total payment? It would not be too hard to work it out by adding up all the individual payments, using the calculator; the more challenging question is whether there is a formula for the total.

To start smaller, what about the sum total over a shorter period than a month. Do you remember the story? Queen Calcula is prepared to give you one gold piece on the first day of the month, two on the second, four (or 2^2) on the third, eight (or 2^3) on the fourth, and so on. How many gold pieces in total has she offered up to any given day?

Table 1 shows the results.

Table 1

Day	Number of gold pieces	Total
1	$1 = 1$	1
2	$1 + 2^1 = 1 + 2$	3
3	$1 + 2^1 + 2^2 = 1 + 2 + 4$	7
4	$1 + 2^1 + 2^2 + 2^3 = 1 + 2 + 4 + 8$	15
5	$1 + 2^1 + 2^2 + 2^3 + 2^4 = 1 + 2 + 4 + 8 + 16$	31

The most important thing to notice about this table is that the entries in the 'Total' column are just short of being powers of 2 (2, 4, 8, 16, 32, ...); in fact, the total on day n (where $n = 1, 2, 3, 4$ or 5) is $2^n - 1$. To see why this is so, consider (for example) the sum for the fifth day:

$$1 + 2^1 + 2^2 + 2^3 + 2^4$$

It contains within it the sum for the fourth day, namely $1 + 2^1 + 2^2 + 2^3$, in *two* ways. Use of brackets to group the terms should make it clear what this means.

The first way is the obvious one: the first four terms of the sum for the fifth day give the sum for the fourth day, thus:

$$(1 + 2^1 + 2^2 + 2^3) + 2^4$$

On the other hand, the *last* four terms of the sum for the fifth day are also closely related to the sum for the fourth day: they give twice that sum, so the sum for the fifth day can also be expressed as:

$$1 + 2(1 + 2^1 + 2^2 + 2^3)$$

To simplify these expressions, write S for the sum for the fourth day. So S just stands for $1 + 2^1 + 2^2 + 2^3$. Then the two expressions for the sum for the fifth day are $S + 2^4$ and $1 + 2S$. These must be equal to each other (since they are merely alternative ways of writing the same thing), so:

$$S + 2^4 = 1 + 2S$$

Treat this as an equation for S, and solve it. To do so, first subtract S from each side of the equation, to leave:

$$2^4 = 1 + S$$

It should now be clear that:

$$S = 2^4 - 1$$

This must seem to you like a roundabout way of finding out something that you knew already from the table. However, the same sort of technique will work for *every* case, not just for those cases that appear in the table. It is feasible, in fact, to go straight to the case raised by the Queen Calcula problem: how many gold pieces in total would you receive from Her Majesty over a 31-day month? In other words, is there a compact expression for the following sum?

$$1 + 2^1 + 2^2 + 2^3 + \cdots + 2^{30}$$

(Note from Table 1 that the power of the final term in the sum is always one less than the number of the last day.)

Perhaps during the last few seconds, while your conscious mind was attending to the remark in brackets, in your subconscious mind you did two things: you might have guessed that the answer to the question posed a moment ago is $2^{31} - 1$; and you realized that the next step is to consider the two comparable ways that the sum $1 + 2^1 + 2^2 + 2^3 + \cdots + 2^{30}$ is contained in the total for the next day, namely:

$$1 + 2^1 + 2^2 + 2^3 + \cdots + 2^{30} + 2^{31}$$

Write $S(31)$ to stand for the unknown sum (with the 31 serving as a reminder that you are no longer thinking about the simple case discussed previously). Then the two expressions $S(31) + 2^{31}$ and $1 + 2S(31)$ are both equal to $1 + 2^1 + 2^2 + 2^3 + \cdots + 2^{30} + 2^{31}$, and are therefore equal to each other. Thus:

$$S(31) + 2^{31} = 1 + 2S(31)$$

As before, subtract $S(31)$ from both sides, giving:

$$2^{31} = 1 + S(31)$$

It follows that, as predicted:

$$S(31) = 2^{31} - 1$$

So you do not have to go through a long addition sum to calculate the total number of gold pieces that Good Queen Whatsername is handing out this month. And you have no doubt realized that the next step is to derive the general formula for $S(n)$, the total up to the nth day; indeed, you may be itching to work it out for yourself.

Activity 13 Be our guest

Find the formula. (A word of warning: notice that the final term in the sum for the nth day is 2^{n-1}, not 2^n.)

Activity 14 Pieces of eight

How many doubloons, or pieces of eight, or *Louis d'or*, in total, does the noble Queen hand out in February?

This is a possibly unexpected piece of mathematics: it involves a nice argument. (It may seem odd, to start with, that to find the total up to any particular day you have to look at the total up to the next day—but the calculation is very neat once this step has been made, and with hindsight you can see that it is actually a very clever step.) And there is a lot more mileage in it, which is yet to come.

There will be many situations where you want to keep a cumulative total of the population sizes from a population which is growing exponentially, as in the case of the Donation of Calcula. For example, chain letters: it is perhaps more interesting to know the total number of people who will have received a letter rather than the number of letters posted in the nth generation. So consider a population which is growing exponentially with growth factor b: what is the cumulative total?

For simplicity, consider first the case where the size of the population in the starting generation is just 1. Then the population sizes in the first few generations are $1, b, b^2, b^3, \ldots$; and the population size in the nth generation is b^{n-1}. Let $S(n)$ denote the sum of all these terms, of which there are n. So $S(n) = 1 + b + b^2 + b^3 + \cdots + b^{n-1}$

Then:

$$S(n) + b^n = 1 + b + b^2 + b^3 + \cdots + b^{n-1} + b^n = 1 + bS(n)$$

and therefore:

$$S(n) + b^n = 1 + bS(n)$$

Regard this as an equation for $S(n)$. To solve it, subtract $S(n)$ from each side to get:

$$b^n = 1 + (b - 1)S(n)$$

Collect terms, which gives:

$$(b - 1)S(n) = b^n - 1$$

This formula is not valid for $b = 1$, but in this case $S(n)$ is just a string of 1s which are easy to add up.

Finally, divide the equation $(b-1)S(n) = b^n - 1$ by $(b-1)$ to obtain the result:

$$S(n) = \frac{b^n - 1}{b - 1}$$

(The case considered earlier has $b = 2$, so the term $b - 1$ in the denominator of the fraction is 1.)

Activity 15 How many were coming from St Ives?

What is the answer to the St Ives riddle? How many were coming *from* St Ives?

It is not difficult to deal with the more general case of a population whose sizes in the first few generations are $a, ab, ab^2, ab^3, \ldots$, and whose size in the nth generation is $ab^{(n-1)}$.

Activity 16 A spot of algebra

The formula in this general case is actually

$$S(n) = a\left(\frac{b^n - 1}{b - 1}\right)$$

Can you explain why this is so?

When b is less than 1 (so that the population is decreasing exponentially), both brackets of the terms in the formula for $S(n)$ are negative; since the quotient of two negative numbers is positive, the result is still positive, as it should be. However, it is usually more convenient in this case to write the formula as:

$$S(n) = a\left(\frac{1 - b^n}{1 - b}\right) \tag{1}$$

The formula for $S(n)$, in the form given in Activity 16, can be useful in financial problems. Consider, for example, a savings account which earns compound interest, compounded annually, as discussed before. In the previous discussion, it was assumed that the money (and the interest it earned) was left in the account undisturbed, to continue to grow. But there is a curious fact about people and money: they sometimes want to spend it. So examine what happens if the person whose account it is withdraws a fixed sum every year. The sum of money may have been invested to provide the person with a pension, or to finance the annual award of a prize, or indeed any one of a number of other possibilities.

Example 4 Compound interest with withdrawals

Suppose, for definiteness, that £5000 is invested at an annual compound interest rate of 5%, and that at the end of every year, immediately after the interest has been credited, a sum of £300 is withdrawn. How much money is there in the account at the end of every year (after the withdrawal)? (What the question is asking for, of course, is a formula which can be used to calculate the balance of the account in any year, without having to follow the history of the account year by year.)

If you consider what happens to the account for the first few years, you can probably identify the general rule by which it changes, and thereby conjecture the required formula.

At the end of the first year (after the withdrawal of the first £300) the balance of the account (in £) is:

$$5000 \times 1.05 - 300$$

At the end of the second year, the balance is this amount, multiplied by 1.05 (for the interest earned), less 300 (for the withdrawal); that is,

$$\begin{aligned}&(5000 \times 1.05 - 300) \times 1.05 - 300\\ &\qquad = 5000 \times (1.05)^2 - (300 + 300 \times 1.05)\end{aligned}$$

At the end of the third year, the balance is:

$$\begin{aligned}&(5000 \times (1.05)^2 - (300 + 300 \times 1.05)) \times 1.05 - 300\\ &\qquad = 5000 \times (1.05)^3 - (300 + 300 \times 1.05 + 300 \times (1.05)^2)\end{aligned}$$

► Can you see what is happening?

At each stage, the 5000 (the initial deposit) gets multiplied by another factor of 1.05, while the expression involving the 300s (the withdrawals) also gets multiplied by 1.05, and another 300 is then subtracted. The term involving 5000 grows according to the usual exponential law; the terms involving 300 go like the sums that have been considered earlier in this section. At the end of the nth year, the balance will be

$$5000 \times (1.05)^n - (300 + 300 \times 1.05 + 300 \times (1.05)^2 + \ldots + 300 \times (1.05)^{n-1})$$

Using the formula for the sum of such expressions, this can be expressed as:

$$5000 \times (1.05)^n - 300 \times \left(\frac{(1.05)^n - 1}{1.05 - 1}\right)$$

This can be rearranged and simplified, using the fact that $300 \div (1.05 - 1) = 300 \div 0.05 = 6000$, to give:

$$5000 \times (1.05)^n - 6000 \times ((1.05)^n - 1) = 6000 - 1000 \times (1.05)^n$$

As you can see, the balance of the account decreases year by year; this is because the amount withdrawn is greater than the amount of interest earned, even in the first year. Using the formula, it is possible to work out

how long it will be until the account is exhausted. As you can check with your calculator, $(1.05)^{36} = 5.8$, while $(1.05)^{37} = 6.1$ (both figures are given to one decimal place). Thus, at the end of the 36th year there is no longer enough money in the account to cover a withdrawal of £300 at the end of the following year.

Activity 17 Another question of money

You decide to invest £1000 in an account which pays compound interest at a fixed annual rate of 7% for five years. How much will you have at the end of the period?

A feature of this account is that you are allowed to withdraw up to £100 at the end of each intermediate year during the five-year term. You decide to use this facility to its maximum extent. What will be the total sum you receive at the end of the five years?

The same approach also covers loans and repayment mortgages: it can be used to find the debt, or amount of money outstanding. If a certain sum is borrowed, there is an annual repayment of a fixed amount, and interest is charged every year at a fixed rate on the sum outstanding. By using similar methods, you can work out what the annual repayment must be to clear the loan in a specified number of years at a given interest rate, and other useful things of that sort.

Here is another application of these ideas in a different context.

Example 5 More about snowflakes

When the snowflake curve was discussed before, the task was to work out a formula for the perimeter of the nth generation figure in the construction. The question now is, what area does this figure enclose?

To tackle this question, the essential point to grasp is that in each generation a certain number of equilateral triangles are attached to the outside of the figure obtained in the previous generation. Thus, to calculate the area, it is necessary to add up the areas of all these triangles. The length of the sides of the new triangles that are added in one generation is one third of the length of the sides of the triangles added in the previous generation. The area of each of these triangles is thus one ninth of the area of each triangle added in the previous generation. So the area of the triangles 'grows' exponentially, with exponential growth factor $\frac{1}{9}$ (so the triangle areas actually *decrease*).

The area of each triangle which is added to the $(n-1)$th generation figure to obtain the nth generation figure is therefore $A \times (\frac{1}{9})^n$, where A is the

area of the original triangle. One small triangle is added to each side of the figure, so the additional area at each generation is the number of sides times the area of one added triangle. There are four times as many sides in one generation as there are in the preceding one, so the number of sides grows exponentially too; the nth generation figure has 3×4^n sides. The area enclosed by the figure grows, generation by generation, as shown in Figure 6.

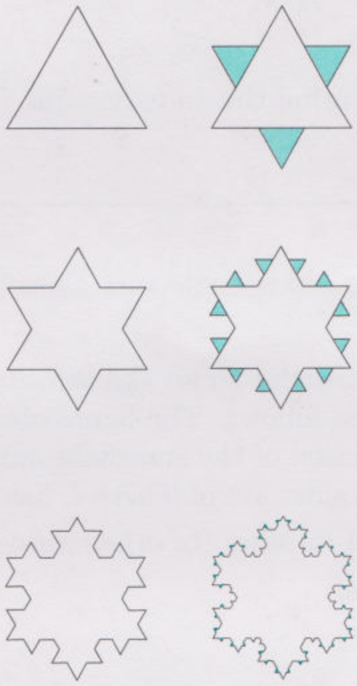


Figure 6 Constructing a snowflake curve: adding triangles

The original triangle has area A.

At the first generation, this is augmented by the addition of a triangle of area $A \times \frac{1}{9}$ to each of the three sides of the original figure, so the area increases to:

$$A + A \times 3 \times \tfrac{1}{9} = A\left(1 + \tfrac{1}{3}\right)$$

In going from the first to the second generation, 3×4 triangles are added, each of area $A \times \left(\frac{1}{9}\right)^2$, so the total area becomes:

$$A\left(1 + \tfrac{1}{3} + 3 \times 4 \times \left(\tfrac{1}{9}\right)^2\right) = A\left(1 + \tfrac{1}{3} + \tfrac{1}{3} \times \tfrac{4}{9}\right)$$

At the next stage, moving on to the third generation, 3×4^2 triangles are added, each of area $A \times \left(\frac{1}{9}\right)^3$, so the total area becomes:

$$A\left(1 + \tfrac{1}{3} + \tfrac{1}{3} \times \tfrac{4}{9} + 3 \times 4^2 \times \left(\tfrac{1}{9}\right)^3\right) = A\left(1 + \tfrac{1}{3} + \tfrac{1}{3} \times \tfrac{4}{9} + \tfrac{1}{3} \times \left(\tfrac{4}{9}\right)^2\right)$$

You can now perhaps see that the area of the nth generation figure is:

$$A\left(1 + \tfrac{1}{3} + \tfrac{1}{3} \times \tfrac{4}{9} + \tfrac{1}{3} \times \left(\tfrac{4}{9}\right)^2 + \cdots + \tfrac{1}{3} \times \left(\tfrac{4}{9}\right)^{n-1}\right) \qquad (2)$$

Now sum all the terms in the bracket except the first:

$$\begin{aligned}&\tfrac{1}{3}+\tfrac{1}{3}\times\tfrac{4}{9}+\tfrac{1}{3}\times\left(\tfrac{4}{9}\right)^2+\cdots+\tfrac{1}{3}\times\left(\tfrac{4}{9}\right)^{n-1}\\ &\quad=\tfrac{1}{3}\left(1+\tfrac{4}{9}+\left(\tfrac{4}{9}\right)^2+\cdots+\left(\tfrac{4}{9}\right)^{n-1}\right)\\ &\quad=\tfrac{1}{3}S(n) \text{ with } b=\tfrac{4}{9}\end{aligned}$$

Using equation (1) on page 26:

$$\tfrac{1}{3}\left(\frac{1-\left(\frac{4}{9}\right)^n}{1-\frac{4}{9}}\right)=\tfrac{3}{5}\left(1-\left(\tfrac{4}{9}\right)^n\right)$$

(since $1-\frac{4}{9}=\frac{5}{9}$). Substituting this in expression (2) gives:

$$A\left(\tfrac{8}{5}-\tfrac{3}{5}\left(\tfrac{4}{9}\right)^n\right)$$

See *Unit 14* for details.

Now the area of an equilateral triangle with sides of length l happens to be $\frac{1}{4}\sqrt{3}l^2$.

So we can summarize the results about the snowflake curve derived in this and the previous section as follows. The figure obtained at the nth generation in the construction of the snowflake curve, starting with an equilateral triangle whose sides are of length l, has the following properties.

◇ The total length of all its sides (in other words, its perimeter) is $l\left(\frac{4}{3}\right)^n$.

◇ The area it encloses is:

$$\tfrac{1}{4}\sqrt{3}l^2\left(\tfrac{8}{5}-\tfrac{3}{5}\left(\tfrac{4}{9}\right)^n\right)$$

There is one more thing to say about the snowflake curve (the most interesting thing, in fact), but that will come in a later section.

The basic principle that was used to calculate the sum:

$$S(n)=1+b+b^2+b^3+\ldots+b^{n-1}$$

makes use of the fact that:

$$1+bS(n)=1+b+b^2+b^3+\ldots+b^{n-1}+b^n$$

But the expression on the right-hand side of this equation is just the sum with one more term; that is to say, it is $S(n+1)$. You have been dealing with a quantity which grows according to the rule:

$$S(n+1)=1+bS(n)$$

It is a combination of the exponential rule and the linear one which was discussed briefly in the example on simple interest. In the more general case, where terms take the form a, ab, ab^2, ...

$$S(n)=a+ab+\ldots ab^{n-1}$$

Then:

$$\begin{aligned}S(n+1)&=a+ab+\cdots+ab^n\\ &=a+bS(n)\end{aligned}$$

So this too is a combination of linear and exponential. There are many situations which lead to models of the form:

$$S(n+1) = a + bS(n)$$

As you have seen already, financial problems involving regular withdrawals from a savings account, loans and mortgages are all of this type. The remainder of this section is devoted to an example which arises in quite a different situation, but which leads to the same kind of model nevertheless.

Example 6 *Irrigation model*

In countries where rainfall is scarce, irrigation of fields is an essential part of agriculture; but supplies of water may be limited, so that unrestricted irrigation is not possible. After each period of irrigation, water is lost from the soil by evaporation. This example deals with a hypothetical irrigation scheme, and examines how the water content of the soil is affected by the alternation of gain through irrigation and loss due to evaporation. Under the scheme, a farmer is allowed to irrigate his or her fields each day from 9 pm to 9 am only; and so farmers apply the same maximum amount of water—the daily ration—to their fields every night. During the day (that is, from 9 am to 9 pm), the strength of the sun is such that half the total water in the topsoil is lost through evaporation. How does the amount of water in the topsoil vary day by day?

Let $w(n)$ be the amount of water in the topsoil at the end of the nth day (a day, for this purpose, being a twenty-four hour period beginning at 9 am). Suppose that irrigation begins after a period of drought when the topsoil is bone dry. Then, on the first day, the fields remain dry until 9 pm; and then during the night one daily ration of water is supplied, so that on the following morning the water content is just 1 (using the constant daily ration as the unit of measurement), since no evaporation occurs from 9 pm to 9 am. Thus:

$$w(1) = 1$$

After twelve hours, evaporation reduces the water content by half, so that the water content reduces to $\frac{1}{2}$ a ration; but in the following twelve hours

another ration of water is supplied, so that the water content goes up to $1 + \frac{1}{2}$; thus:

$$w(2) = 1 + \tfrac{1}{2}$$

In the next twelve hours, the water content reduces by half to $\frac{1}{2} \times \left(1 + \frac{1}{2}\right) = \frac{1}{2} + \left(\frac{1}{2}\right)^2$, and then a further ration is added, to give:

$$w(3) = 1 + \tfrac{1}{2} + \left(\tfrac{1}{2}\right)^2$$

Continuing in the same way, you find that:

$$w(n) = 1 + \tfrac{1}{2} + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^3 + \cdots + \left(\tfrac{1}{2}\right)^{n-1}$$

Using the formula for the sum you end up with:

$$w(n) = \frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} = 2\left(1 - \left(\tfrac{1}{2}\right)^n\right) = 2 - 2 \times \left(\tfrac{1}{2}\right)^n = 2 - \left(\tfrac{1}{2}\right)^{n-1}$$

Now as time passes the powers of $\frac{1}{2}$ become extremely small; for example, taking $n = 14$:

$$\left(\tfrac{1}{2}\right)^{13} = 1.22 \times 10^{-4} \text{ and } w(14) = 1.999\,878$$

It follows that after a couple of weeks the water content in the topsoil is very nearly 2. This means that even when starting with completely dry topsoil, the irrigation procedure quickly ensures that the water content in the topsoil at 9 am each morning the water content is *double* the daily ration. The irrigation scheme therefore reaches a 'steady state'. Each morning at 9 am there is a double ration of water in the soil. This evaporates down by half to one by 9 pm, when irrigation starts, restoring the water content over the next twelve hours, and the cycle is then repeated. If the soil starts off with some moisture in it, then the transition to the same steady state is more rapid.

Note that the relation governing the amount of water in the soil is:

$$w(n+1) = 1 + \tfrac{1}{2}w(n)$$

Activity 18 *If the soil isn't dry to begin with ...*

Suppose that the farmer starts off, not with dry soil, but with soil that already contains half a day's ration, so that $w(0) = \frac{1}{2}$. How does this affect the result of the irrigation programme?

In this section, you have met several situations in which a population grows exponentially, where interest is not just in the size of the population in any one generation, but the accumulation over several generations.

There is a useful formula for this accumulation:

$$S(n) = a\left(\frac{b^n - 1}{b - 1}\right)$$

Outcomes

After studying this section, you should be able to:

◇ decide, in a given situation involving exponential growth, when it is appropriate to calculate the accumulation over several generations rather than just the population size for any given stage (Activity 17);

◇ understand the derivation of the formula for the accumulation (Activities 13, 16);

◇ calculate the accumulation, where appropriate, referring to and using the formula (Activities 14, 15, 18).

3 Exponential graphs

Aims This section aims to help you become familiar with the graphs of exponential functions. ◇

In the previous two sections, a number of models have been developed which involve exponential functions, either directly (in the examples in Section 1) or indirectly (in most of the examples in Section 2). This section takes up the suggestion, made at the end of Subsection 1.2, that it would be worth investigating the properties of these models by looking at the graphs of the exponentials involved.

A few words of clarification of terminology are in order before going any further. One of the examples in Section 1 led to the formula:

$$P(n) = 200(1.1)^n$$

(It is actually the formula for the the amount of money in a savings account, but this is not important for present purposes.) As you will recall, when using a calculator to graph something, you have to use the standard symbols for the independent and dependent variables which the calculator imposes. So the formula above must be converted into a standard form, which is done by replacing n by x, and $P(n)$ (which started off life as plain P before it got expanded for emphasis) by y. When this is done, you obtain:

$$y = 200(1.1)^x$$

This specifies a function, an exponential function.

An exponential function is one of the form:

$$y = ab^x$$

where a and b are numerical constants (200 and 1.1, respectively, in the particular example).

In Example 6, the following formula was obtained:

$$w(n) = 2 - \left(\tfrac{1}{2}\right)^{n-1} = 2 - 2\left(\tfrac{1}{2}\right)^n$$

Replacing n by x and $w(n)$ by y to convert it into the standard form that the calculator can understand leads to:

$$y = 2 - 2\left(\tfrac{1}{2}\right)^x$$

This is not quite an exponential function; rather than being of the simple exponential form it is an example of a function of the form:

$$y = ab^x + c$$

Here c is another constant. In the particular example,

$$a = -2, \qquad b = \tfrac{1}{2}, \qquad \text{and} \qquad c = 2$$

Activity 19 Conversion of formulas

Here are some of the formulas that were derived in the previous two sections. Convert each of them into the standard form, by suitable substitutions of x and y. Say in each case whether the function is a plain exponential or an example of the variant $y = ab^x + c$, and identify the particular values of a, b, and (if appropriate) c.

(a) $P(n) = 3 \times \left(\frac{4}{3}\right)^n$

(b) $P(n) = l\left(\frac{4}{3}\right)^n$

(c) $P(n) = 5^n$

(d) $S(n) = 2^n - 1$

(e) $B(n) = 6000 - 1000 \times (1.05)^n$

(In the last of these, there was no explicit $B(n)$ formula in the text. Think of B as standing for 'balance' (another of the financial examples), if you like.)

All this suggests that it would be worth looking at the graphs of $y = ab^x$ and of $y = ab^x + c$.

The rest of this section consists of a sequence of exercises in which you are invited to investigate the shapes and properties of these families of graphs of exponential functions, using your calculator. Before you begin, you should refer to Section 12.1 of the *Calculator Book*, which covers all the information about entering, tabulating and graphing exponential functions which you will need.

Now study Section 12.1 of Chapter 12 in the Calculator Book

Activity 20 An exponential graph

Draw the graph of the function $y = 3^x$ on your calculator, for x in the range -3 to 3 and y in the range 0 to 28.

Describe the general shape of the graph. You might try to compare the graph with a straight-line graph and with a parabola—in which ways (if any) is the exponential similar to either of these? In which ways is it different?

The graph of $y = 3^x$ goes shooting up as you move to the right, but hugs the x-axis more and more closely as you go to the left. These observations can be confirmed by considering the numerical values of the function.

Activity 21 Numerical values

Consult a table of values of $y = 3^x$ to answer the following questions.

(a) What is the value of y when $x = 0$? When $x = 1$? When $x = -1$?

(b) Describe how the values of 3^x change as x gets bigger and bigger (increases), starting at $x = 0$; and as x decreases (that is, becomes more negative), starting at $x = 0$.

Take the idea of comparing linear, quadratic and exponential graphs more seriously. From part (a) of Activity 21, you know that the exponential $y = 3^x$ satisfies $y = 1$ when $x = 0$, and $y = 3$ when $x = 1$. Here is the equation of a straight line that also satisfies these conditions:

$$y = 2x + 1$$

Here is the equation of a parabola, which does the same thing:

$$y = 2x^2 + 1$$

Activity 22 Making comparisons

Graph the equations producing the straight line and the parabola simultaneously with the exponential. Write down one significant similarity between the exponential graph and the straight line, and one significant difference (not just that the exponential is not straight—you know that already). Write down one significant similarity between the exponential graph and the parabola, and one significant difference.

As a result of this exercise in comparative graphics, you should never mistake an exponential graph for a linear or a quadratic one.

There is a useful bit of terminology to describe the fact that 3^x gets closer and closer to zero as x gets more and more negative: we say that 3^x *tends to* 0 *as x tends to minus infinity* $(-\infty)$.

(It is curiously difficult to find a clear and unambiguous form of words to describe a progression like $-5, -10, -15, -20, \ldots$. You clearly cannot say that x is getting bigger and bigger, because it is not; bigger and bigger means moving to the right along the number-line, not the left. But, on the other hand, you cannot say smaller and smaller, because in ordinary English this usually means 'closer and closer to 0'. The phrase 'more and more negative' looks a bit strange. So some mathematical terminology was created for this part of the claim, and that is the role of the phrase 'x tends to minus infinity'.)

The behaviour of the graph, and of the values of 3^x, as x gets bigger and bigger, can be expressed in similar terms: say here that 3^x *tends to plus infinity* $(+\infty)$ *as* x *tends to plus infinity* $(+\infty)$. (The plus sign is optional here, just as it is optional in front of a positive number; it is included to draw attention to the contrast between this and the previous case.)

Activity 23 Tendencies

Describe the behaviour of $y = 2x + 1$ as x tends to $+\infty$ and as x tends to $-\infty$ (your answer should take the form 'as x tends to $+\infty$, $2x + 1$ tends to ...'). Do the same for $y = 2x^2 + 1$.

Again, if you can remember these different 'tendencies', you should never confuse the graphs of the three types of formula.

So far, you have looked at just one exponential graph. The next step is to compare the graphs of different exponentials.

Activity 24 Some more exponential graphs

Graph $y = 3^x$, $y = 4^x$ and $y = 5^x$ on the same screen. Use the same x-range as before, and adjust the y-range so as to get all the graphs on the screen.

(a) For each of the new graphs, what is the value of y when $x = 0$ and when $x = 1$?

(b) Describe the (common) general shape of all three graphs. How do the three graphs differ? Where do you think the graph of $y = 6^x$ would lie, in relation to the others? What about $y = 2^x$? And what about $y = (3.5)^x$? Work out the answer for yourself first, and then confirm by using the calculator.

(c) Can you hazard a guess about the general shape of the graph of $y = b^x$, where b is some number greater than 1? How would the graphs of $y = b^x$ and $y = B^x$ compare, if b and B are both numbers greater than 1, and $B > b$?

Activity 25 Yet more exponential graphs

Clear the screen of your calculator, and graph the following equations, one at a time, and then all together: $y = (0.5)^x$; $y = (0.25)^x$; $y = (0.125)^x$. Use the same x-range as before, and adjust the y-range to get all the graphs on the screen.

(a) What is the value of y when $x = 0$, in each case? When $x = 1$? When $x = -1$?

(b) Describe the (common) general shape of all three graphs. How do the three graphs differ? Where do you think the graph of $y = (0.6)^x$ would

lie, in relation to the others? What about $y = (0.2)^x$? And what about $y = (0.025)^x$? Work out the answer for yourself first, and then confirm by using the calculator.

(c) Can you hazard a guess as to the general shape of the graph of $y = b^x$, where b is some positive number less than 1? What do you think b^x tends to as x tends to $+\infty$ or to $-\infty$ in this case? How would the graphs of $y = b^x$ and $y = B^x$ compare, if b and B are both positive numbers less than 1, and $B > b$?

(d) Is there any relation between the graphs of equations of the form $y = B^x$, where $B > 1$, and the graphs of equations of the form $y = b^x$ where b is a positive number less than 1? If so, what is the relation?

So you now know what the graph of the function $y = b^x$ looks like, in all cases of interest except one: the case $b = 1$ (though when you have worked out what this graph looks like, you may think that it was not a very interesting case after all).

Activity 26 A common point of exponential graphs

(a) Describe the graph of $y = 1^x$. You may use your calculator if you really want to.

(b) There is just one point which *every* exponential graph $y = b^x$ passes through: what is it?

So far, of the particular kinds of exponential function: $y = ab^x$, you have considered only graphs of functions like $y = b^x$; that is, those for which $a = 1$. There is therefore one more thing to be done: find out what effect the value of the parameter a has on the graph.

Activity 27 Exponential graphs once more

Graph the function $y = 2^x$ on your calculator, using the same x-range as before. Then superimpose, one at a time, the graphs of $y = 3 \times 2^x$; $y = 4 \times 2^x$; and $y = 0.3 \times 2^x$.

(a) What is the value of y when $x = 0$, in each case?

(b) Describe the similarities and the differences between these graphs.

Activity 28 One more time

Graph the function $y = -2^x$ (that is, $y = (-1) \times 2^x$). (You will have to reset the y-range, or you will not see anything!)

Turn your attention now to graphs of functions of the form $y = ab^x + c$.

Activity 29 Variations on the theme

(a) Can you predict how the graph of $y = 2^x - 1$ is related to the graph of $y = 2^x$? (The relationship is the same as that between $y = x^2 - 1$ and $y = x^2$, which you investigated in *Unit 11*.) Check by graphing both functions.

(b) Graph the function $y = 6 - (1.05)^x$, for x between 0 and 50, and for y between -5 and 5. (This is more or less the graph of the function which models the balance in a savings account from which £300 is withdrawn every year. The balance has been expressed in multiples of £1000 to save having to enter a lot of 0s into the calculator.) Find when the balance reduces to £0.

One class of functions consists of those of the form $y = ab^x + c$ for which b is less than 1. Several examples have occurred in the previous section, including $y = 2 - 2 \times \left(\frac{1}{2}\right)^x$, which arose in the model of irrigation. The text drew your attention to the fact that the value of the water content (denoted now by y) very quickly gets very close to 2, and stays there. This behaviour is very clearly shown by the graph.

Activity 30 The irrigation model

Graph $y = 2 - 2 \times \left(\frac{1}{2}\right)^x$ for x between 0 and 10. When you have done so, you may find it instructive to repeat this, this time including the graph of $y = 2$ as well.

These graphing activities illustrate an important property of functions of the form $y = ab^x + c$ for which b is less than 1; in such a case, $ab^x + c$ tends to c as x tends to $+\infty$. It is easy to see why this must be so: b^x tends to 0 as x tends to $+\infty$. This kind of function, therefore, can be used to model something which grows (or decays) steadily to some non-zero limit. The simple exponential model $y = ab^x$ can describe only two kinds of long-term behaviour: unlimited growth ($b > 1$); or decline to zero ($b < 1$).

You might like to update your Handbook sheet on the functions $y = ab^x$ and $y = ab^x + c$.

This is a convenient point to make a final comment about the snowflake curve. But first you must turn your mind away from exponential graphs for a moment, call up your geometric intuition, and be prepared to give it free rein.

A circle, a triangle, and a square are all examples of closed figures—figures which come back to where they begin. Each of them encloses a certain area, and each of them has a certain length. You will be able to think of many other figures with these properties.

Formulas for working out areas of different, closed geometric figures are explored in *Unit 14*.

On the other hand, a parabola (extended arbitrarily far in both directions), and a straight line (likewise), are both examples of curves which have infinite length—but neither of them encloses an area. Now, here comes the question: can there be a closed curve, which (like the circle) encloses a finite area, and yet (like the parabola) has infinite length? You think so? Can you imagine it?

Recall that the figure obtained at the nth generation in the construction of the snowflake curve, starting with an equilateral triangle whose sides are of length l, has the following properties.

- ◇ The total length of all its sides (in other words, its perimeter) is $l\left(\frac{4}{3}\right)^n$.
- ◇ The area it encloses is:

$$\tfrac{1}{4}\sqrt{3}l^2\left(\tfrac{8}{5}-\tfrac{3}{5}\left(\tfrac{4}{9}\right)^n\right)$$

Now these data are for the figures that are constructed *en route* to the ultimate snowflake curve; these figures are just finite approximations to the real thing. To find the length of the perimeter of the snowflake curve itself, and the area it encloses, it is necessary to examine what happens to the expressions above as n tends to $+\infty$.

The snowflake curve is an example of a *fractal* which is discussed in the TV programme *The True Geometry of Nature.*

The formula for the length is of the form ab^n where b is *greater* than 1; the length therefore increases indefinitely, and so the perimeter of the snowflake curve is infinitely long. The formula for the area, on the other hand, is of the form $ab^n + c$ with b *less* than 1; so the area enclosed by the snowflake curve is just c; that is, $\frac{1}{4}\sqrt{3}l^2 \times \frac{8}{5}$, or $\frac{2}{5}\sqrt{3}l^2$, or $\frac{8}{5}$ times the area of the original triangle. The snowflake curve is therefore infinitely long, despite the fact that it is a closed curve which encloses a finite area.

You could add some notes to your Handbook sheet about the snowflake curve.

Outcomes

After studying this section, you should be able to:

- ◇ describe the characteristic features of the graph of an exponential function (Activities 20, 21);
- ◇ recognize exponential graphs, and in particular distinguish them from graphs of other classes of functions such as linear functions and quadratics (Activities 22, 23);
- ◇ describe how the values of the constants a and b affect the shape and position of the graph of $y = ab^x$ (Activities 24, 25, 26, 27, 28);
- ◇ describe the graph of the modified exponential function $y = ab^x + c$ (Activity 29);
- ◇ draw conclusions about the long-term behaviour of the accumulation of a population when the growth factor of the population is less than 1 (Activity 30).

4 Exponents

Aims This section aims to introduce some important rules for operating with and simplifying expressions involving exponents. ◇

How are you getting on with seeing, saying and recording as you meet new ideas?

The graphical investigations of the previous section have brought to light some interesting features of graphs of functions of the form $y = ab^x$. Not the least of these is the fact that there is a graph at all: that ab^x has a value when x is not a whole number. In this section, you will see some properties of powers (exponents) which will explain this feature of the graph.

4.1 Rules for exponents

A kilometre is a thousand metres; a kilogram is a thousand grams. The prefix 'kilo-' means a thousand, and putting 'kilo' in front of the name of a unit produces the name of a new unit which is one thousand of the previous ones. This is a principle that could be applied over and over again: you might call a thousand kilometres a kilokilometre, and a thousand kilokilometres a kilokilokilometre. But in fact, there are special names for these large units: a kilokilo(unit) is called a mega(unit), and a kilokilokilo(unit) is a giga(unit).

▶ How many metres are there in a gigametre?

A kilometre is 1000 metres, or 10^3 metres, so a megametre is $1000 \times 1000 = 1\,000\,000$ metres, or 10^6 metres, and a gigametre is $1000 \times 1\,000\,000 = 1\,000\,000\,000$ metres, or 10^9 metres. Notice how much more convenient it is to write these large numbers in exponential (or scientific) form than in ordinary decimal form. This is partly because exponential form saves space, of course: but there is more to it than that. There is a simple pattern to the way that the exponent changes each time the number is multiplied by 1000, or 10^3: it just increases by 3.

▶ A tera(unit) is 1000 giga(unit)s. How many metres are there in a terametre? How many metres are there in 100 terametres?

There are 10^{12} metres or 10^9 kilometres in a terametre. There are 100×10^{12}, or $10^2 \times 10^{12}$, metres in 100 terametres: that is, 10^{14} metres.

It is useful to express the relations between these units in terms of powers of 10 because when two powers of 10 are multiplied together, their exponents combine in a particularly simple fashion.

> The exponent of the product is the sum of the exponents of the factors.

The discussion of units above has made use of the following cases of this generalization:

$$\begin{aligned}
10^3 \times 10^3 &= 10^{3+3} &&= 10^6\\
10^3 \times 10^6 &= 10^{3+6} &&= 10^9\\
10^3 \times 10^9 &= 10^{3+9} &&= 10^{12}\\
10^2 \times 10^{12} &= 10^{2+12} &&= 10^{14}
\end{aligned}$$

This is a general rule; that is to say, it holds whenever multiplying two powers of 10 together (or indeed, as you will see later, when multiplying two powers of any number together). So far as it applies to powers of 10 it is a simple rule, and, it is to be hoped, a familiar one—though you may not have thought about it before in just the way it has been expressed here.

Activity 31 Some other ways of looking at it

(a) Think about multiplying together two numbers of the form 100 . . . 000: for example,

$$10\,000\,000 \times 100\,000\,000\,000$$

How can you tell how many zeros there will be following the 1 in the answer?

(b) A million is a thousand thousands; a billion is a thousand millions. Express as powers of 10: ten billion millions; one hundred billion billions; one million billion thousand million billions.

The rules for exponents: multiplication

The basic rule for multiplying powers of 10 is as follows.

> The exponent of a product of powers of 10 is the *sum* of the exponents.

In symbols:

$$10^m \times 10^n = 10^{(m+n)}$$

Now it is all very well just *saying* that $10^m \times 10^n = 10^{(m+n)}$, but how do you know it is really true? You have worked it out for $10^3 \times 10^6 = 10^9$, and so on: but it is just possible that that was a coincidence, or a stroke of luck, or that some examples were carefully chosen which happen to work

out like that. But, from the following you can see it is *always* going to work out like that. First, think about $10^3 \times 10^6 = 10^9$:

$$\begin{aligned}10^3 \times 10^6 &= (10 \times 10 \times 10) \times (10 \times 10 \times 10 \times 10 \times 10 \times 10)\\ &= 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10\\ &= 10^9\end{aligned}$$

all you have to do is count the total number of 10s. And that is how it will work always.

$$\begin{aligned}10^m \times 10^n &= \underbrace{(10 \times 10 \times \ldots \times 10)}_{m \text{ factors}} \times \underbrace{(10 \times 10 \times \ldots \times 10)}_{n \text{ factors}}\\ &= \underbrace{10 \times 10 \times \ldots \times 10}_{(m+n) \text{ factors}}\\ &= 10^{m+n}\end{aligned}$$

This rule is commonly used when multiplying numbers expressed in scientific notation. For example: the distance from the Earth to the Sun is (approximately) 1.5×10^8 kilometres. To express this in metres you merely have to multiply it by 10^3, and to do that merely add 3 to the exponent 8, to obtain 1.5×10^{11}. Again: the speed of light is about 3×10^5 kilometres per second, and there are approximately 3×10^7 seconds in a year; so in a year, light travels $(3 \times 10^5) \times (3 \times 10^7)$ kilometres. To evaluate this product, multiply the two factors 3 together, and add the exponents 5 and 7; so the distance light travels in one year is about 9×10^{12} kilometres. (This distance is usually called a *light-year*.)

Activity 32 Inching to the Sun

(a) The distance from the Earth to the Sun is 1.5×10^8 kilometres. There are about 3.9×10^4 inches in a kilometre. What is the distance from the Earth to the Sun in inches?

(b) The distance of Pluto, the furthest planet, from the Sun is about 5.9×10^9 kilometres. How far is it from Pluto to the Sun in inches?

In all the cases used so far to illustrate the rule for multiplying powers of 10, namely $10^m \times 10^n = 10^{(m+n)}$, the exponents m and n have been positive whole numbers. One reason why the rule is so important is that it holds in lots of other cases as well—in fact, it holds for any numbers m and n whatsoever.

▶ But just what is meant by the expression 10^m when m is not a positive whole number? What does 10^{-5} mean, for example? What about $10^{-2/3}$? And 10^0?

You may already have some answers to some or all of these questions. The point to emphasize here is that the meanings of such expressions are consistent with the rule for multiplying powers—more than that: when you know the rule, you can work out the meanings.

Here is an instance: What would 10^0 have to be in order to be consistent with the rule for multiplying powers? If the rule $10^m \times 10^n = 10^{(m+n)}$ is to hold when n (say) is 0, then it must be the case that:

$$10^m \times 10^0 = 10^{(m+0)} = 10^m$$

So, 10^0 must be a number such that multiplying by it does not change anything (at least so long as the number being multiplied is a power of 10). There is only one number which has no effect when you multiply by it, and that number is 1; so 10^0 must be 1.

So much for multiplying powers of 10, for the moment. What happens when dividing one power of 10 by another?

▶ How many kilometres are there in a gigametre?

A gigametre is 10^9 metres, while a kilometre is 10^3 metres. To find how many kilometres there are in a gigametre, divide 10^9 by 10^3. To do this, multiply together 9 factors of 10, and then strike out 3 of them, leaving 6 factors of 10; the result is 10^6 which is $10^{(9-3)}$.

When dividing one power of 10 by another, the result is another power of 10; to calculate the exponent of the answer, all you have to do is subtract the exponent of the divisor from the exponent of the number into which it is divided.

Activity 33 *Another way of looking at it*

Think about dividing one number of the form 100...000 by another: for example,

10 000 000 000 ÷ 100 000

How could you tell how many zeros there will be following the 1 in the answer?

The rule for dividing one power of 10 by another is at its most obvious when a smaller number is divided into a larger one, so that the result is a number greater than 1. When this is the case, the exponent of the divisor is smaller than the exponent of the number which is being divided, so their difference is positive, and the answer is a power of 10 with a positive exponent. This is what happens when, for example, 10^9 is divided by 10^3, to get $10^{(9-3)} = 10^6$. How, if at all, does this fit in with the rule for multiplying powers? A first reaction may be to say 'not at all', because attention has switched from multiplication to division. But division is closely bound up with multiplication, and this relationship is reflected in the rule for powers.

The connection comes via the use of negative exponents. As you probably know, it is common to write (for example) 10^{-3} for 0.001, which can also

be written as $1/10^3$. In general, 10^{-m} is just the reciprocal of 10^m. This must hold true to be consistent with the rule for powers. Consider what the rule, $10^m \times 10^n = 10^{(m+n)}$, implies when n has the value $-m$:

$$10^m \times 10^{-m} = 10^{(m+(-m))} = 10^0 = 1$$

This means that 10^{-m} is the reciprocal of 10^m, as was just claimed.

Now dividing by a number is just the same as multiplying by its reciprocal. Using the example of dividing 10^9 by 10^3 again, see that:

$$10^9 \div 10^3 = 10^9 \times \frac{1}{10^3} = 10^9 \times 10^{-3}$$

The rule for powers as before gives:

$$10^9 \times 10^{-3} = 10^{(9+(-3))} = 10^{(9-3)} = 10^6$$

In fact, the rule $10^m \times 10^n = 10^{(m+n)}$ holds whether m and n are positive, negative or zero. Although called a rule for *multiplying* powers, it covers division as well as multiplication, by making use of the fact that division of one number by another is the same as multiplication of the first number by the reciprocal of the second.

The rules for exponents: powers

The calculations of the numbers of metres in a megametre, a gigametre and a terametre could have been presented in a slightly different fashion, which illustrates another rule for calculations with powers of 10. A megametre is a kilokilometre, so to find how many metres it is calculate $10^3 \times 10^3$; the result, as you have seen, is 10^6. Now multiplying 10^3 by itself is just squaring it; so this calculation could be written as $(10^3)^2 = 10^{(3\times 2)}$. In a similar vein, observe that a gigametre is a kilokilokilometre, so to find the equivalent in metres this time, calculate $10^3 \times 10^3 \times 10^3$, which is 10^3 itself cubed, or $(10^3)^3$. There are 10^9 metres in a gigametre; this result could be written as $(10^3)^3 = 10^{(3\times 3)}$. The fact that a terametre is 10^{12} metres illustrates that $(10^3)^4 = 10^{(3\times 4)}$.

Here, again, there is a general rule.

> The exponent of a power of a power of ten is the *product* of the exponents.

In symbols:

$$(10^m)^n = 10^{(m\times n)}$$

This rule, like the one for multiplication of powers of ten, holds whether m and n are positive, negative or zero.

The rules for exponents may be illustrated, and checked, using a calculator. There is a sequence of exercises with this purpose in Chapter 12 of the *Calculator Book*.

Section 12.2 will be studied in Section 6.

Now study Section 12.3 of Chapter 12 in the Calculator Book.

4.2 Fractional exponents

The last subsection dealt with exponents which are whole numbers, whether positive, negative or zero. It might look like the end of the story: but it is not by a long way. The reason is that it happens to be possible to make sense of 10^r when r is not a whole number, but a fraction—any fraction at all.

The rule $(10^m)^n = 10^{(m \times n)}$ gives the clue. Take a simple example to begin with: $10^{1/2}$. Whatever this may mean, it is reasonable to hope that the same rules continue to apply to it. If this is so, there is one particular choice of m and n which reveals what meaning it *must* have in order to be consistent with these rules: $m = \frac{1}{2}$ and $n = 2$. Substituting these values gives:

$$\left(10^{1/2}\right)^2 = 10^{(1/2 \times 2)} = 10^1 = 10$$

So when $10^{1/2}$ is squared, the result is 10. In other words,

$$10^{1/2} = \sqrt{10}$$

That is to say, the 'one-half power' of 10 must be interpreted to mean the square root of 10.

You probably will be unsurprised to be told next that the 'one-third power' of 10 is the cube root of 10, the 'one-quarter power' of 10 is the fourth root of 10, and so on. It is easy to see why these must be so, and to deal with the general case, by using the rule for exponents of powers again:

$$(10^{1/n})^n = 10^{(1/n \times n)} = 10^1 = 10$$

So:

Recall, in *Unit 9*, the twelfth root of $\frac{1}{2}$ was sometimes written as $\left(\frac{1}{2}\right)^{1/12}$.

$$10^{1/n} = \sqrt[n]{10}$$

That takes care of fractional exponents of the form 'one over something'.

▶ What about more general fractions, such as $\frac{3}{4}$?

Well, $\frac{3}{4} = \frac{1}{4} \times 3$; and once again the rule for exponents of powers determines how to interpret $10^{3/4}$:

$$10^{3/4} = 10^{(1/4 \times 3)} = (10^{1/4})^3 = (\sqrt[4]{10})^3$$

So to calculate $10^{3/4}$, take the fourth root of 10 and cube the result. And the same holds for any fractional exponent:

$$10^{m/n} = (10^{1/n})^m = (\sqrt[n]{10})^m$$

This entire section has so far been concerned exclusively with powers of 10. It is, of course, an accident that our arithmetic is based on the number 10; had humans been endowed with more or fewer fingers, we would perhaps have used a number system with a different base (and indeed there are vestiges of different bases around, such as the use of 60 in the measurement of times and angles). It is certainly the case that there is nothing special about the number 10 so far as the rules for exponents are

concerned: indeed, they hold for powers of any positive number—it does not even have to be a whole number. The argument that was used to show that $10^m \times 10^n = 10^{(m+n)}$ can easily be adapted to show that $b^m \times b^n = b^{(m+n)}$, as follows:

You may wish to recall *Unit 9*, and the work you did on equal temperament, from the point of view of operating with powers of $r = \left(\frac{1}{2}\right)^{1/12}$ and $s = (2)^{1/12}$.

$$\begin{aligned} b^m \times b^n &= \underbrace{(b \times b \times \cdots \times b)}_{m \text{ factors}} \times \underbrace{(b \times b \times \cdots \times b)}_{n \text{ factors}} \\ &= \underbrace{b \times b \times \cdots \times b}_{(m+n) \text{ factors}} \\ &= b^{(m+n)} \end{aligned}$$

It is also true that $(b^m)^n = b^{(m \times n)}$. The rules about negative and fractional exponents follow: $b^0 = 1$, b^{-1} is the reciprocal of b, and $b^{1/n}$ is the nth root of b.

It may have occurred to you that there was actually a choice in how to calculate $10^{3/4}$. We expressed $\frac{3}{4}$ as $\frac{1}{4} \times 3$, and deduced that $10^{3/4} = (10^{1/4})^3 = (\sqrt[4]{10})^3$. But you could just as well have said $\frac{3}{4} = 3 \times \frac{1}{4}$, which would have given:

$$10^{3/4} = 10^{(3 \times 1/4)} = (10^3)^{1/4} = \sqrt[4]{10^3}$$

instead of $(\sqrt[4]{10})^3$. This would imply that to calculate $10^{3/4}$ you cube 10 first and take the fourth root of the result, rather than take the fourth root of 10 first and then cube the result. Fortunately, it does not matter which order you do these two operations in—cube first and then take the fourth root, or take the fourth root first and then cube—you get the same answer either way. This, too, is true whatever the number b:

$$b^{m/n} = (\sqrt[n]{b})^m = \sqrt[n]{(b^m)}$$

You can use arithmetic of fractions to discover some quite complicated things about roots: for example, since $\frac{3}{9} = \frac{1}{3}$:

$$\sqrt[9]{b^3} = \sqrt[3]{b}$$

Activity 34 Check

Use your calculator to find $10^{0.75}$; check that the fourth power of your answer is 1000.

You may omit this activity if you are short of time.

Activity 35 The well tempered clavier

The frequency of a note one octave above a given note is double the frequency of the lower note. In an equally tempered scale, the twelve semitones which make up an octave have the property that the frequency of any note is a fixed multiple of the frequency of the note a semitone below it. You know from *Unit 9* that the multiplier which relates the frequency of a note to that of the note one semitone above is $2^{1/12}$.

One way of writing the chromatic scale in an octave beginning with C is

C, C♯, D, D♯, E, F, F♯, G, G♯, A, A♯, B, C

Show that the frequency of any note C is a factor $1/(\sqrt[4]{2})^3$ of the frequency of the A next above it. Remember how in *Unit 9* 'concert pitch' is defined by setting the frequency of the A above middle C to be 440 cycles per second. Use this result to calculate the concert pitch frequency for middle C and check your answer with what you found in the activity in *Frame 2* in Section 3 of *Unit 9*.

One result of all your work in this section is that you should now know what your calculator is actually calculating when you enter (say) $17.43^{-2.25}$; and you should now understand better why, when you set it to graph $y = 2^x$, you get a curve, extending to the negative x-axis, and not just a collection of isolated points corresponding to the positive whole number values of x.

4.3 Using the rules for exponents

Suppose you send out a chain letter, in which each person who receives a letter is asked to send out 5 copies; suppose further that everyone complies, and that all the letters get sent to different people. How many letters are sent out at, say, the 100th generation? That is easy: 5^{100}. Now this is a perfectly precise way of giving the answer, but it is perhaps not a terribly informative one, if you want to get some idea just how many letters there are—more than the population of London? More than the population of Scotland? More than the population of Europe? These numbers are given as decimal numbers, not as powers of 5. Since they are quite large numbers you might write them in scientific notation: the population of London, for example, is about 7×10^6. So how does 5^{100} compare with 7×10^6?

You could answer this question simply enough by evaluating 5^{100} using your calculator. But spare a thought for those unfortunates of previous generations who did not have calculators: how could they deal with a problem like this? Not by multiplying 5 by itself 100 times by hand, presumably. Well, someone might have been prepared to do a calculation like this once; but if you think about it for a moment you will see that there are many many interesting questions of the form: 'How big is something to the power of something else?' And once you have thought of all those, you will think of some variations on the theme: 'How many generations of the chain letter are required for the number of letters sent to exceed the population of London?', for example. Surely grinding away multiplying $5 \times 5 \times 5 \times \ldots$, or $42 \times 42 \times 42 \times \ldots$, or whatever is not the only way of solving such problems.

This is a type of question that could be tackled systematically, without a calculator (though you would need a book of numerical tables, as will be explained later). To see how, it will be necessary to apply some of the

rules which were derived in the previous section. The question is being investigated not just out of antiquarian interest, but because it introduces some ideas which are important and useful even in the calculator age.

Consider the specific example with which the subsection started, which amounts to the following: express 5^{100} in scientific notation—in other words, in the form $a \times 10^n$ (where a is a number between 1 and 10, and n is a whole number, both of which are to be found). But in considering this problem, bear it in mind that it is being taken as one example of a general type of problem, and that the method which will be described will apply to all problems of this general type.

The key step is this: suppose you knew how to express the number 5 as a power of 10, that is, suppose you could find a number x such that $10^x = 5$; then you could write, using the rule for the exponent of a power:

$$5^{100} = (10^x)^{100} = 10^{100x}$$

and the problem of expressing 5^{100} in scientific notation has been more or less reduced to multiplication (to multiplying this value of x by 100).

▶ Is there a number x such that $10^x = 5$? If you think there is, how could you set about finding it, using your calculator?

One possible way of answering this question is to think of $10^x = 5$ as an equation to be solved to give the required number, x. The question then becomes:

▶ Can this equation be solved, and if so, how?

Here, again, the advantages of living in the calculator age become apparent. In order to try to solve such an equation, all you have to do is graph the function $y = 10^x$, and find whether the graph crosses the line $y = 5$, and if so, where.

Activity 36 *Solving the equation*

Use your calculator to solve the equation $10^x = 5$ graphically, giving your answer correct to four decimal places.

So there certainly is a number x such that $10^x = 5$: the evidence which clinches this claim is that the graph of the function $y = 10^x$ does cross the line $y = 5$. The value of x is 0.6990, accurate to four decimal places.

Activity 37 *Check*

Check that $10^{0.6990}$ is a reasonably good approximation to 5, by calculating it directly (using your calculator). It is not *exactly* 5, because 0.6990 is not

the exact solution to the equation $10^x = 5$. The best way to think about the process of solving the equation graphically gives you a succession of approximate solutions which get closer and closer to ('tend to' in the language of the last section) the exact solution, in the sense that the corresponding values of 10^x get closer and closer to (tend to) 5. To see this process in action, try calculating the values of 10^x when $x = 0.7$, 0.699, 0.698 97, 0.698 970 004 (these are successive approximations to the solution).

Activity 38 What does it all mean?

Remind yourself exactly what $10^{0.7}$ means (in terms of powers of roots of ten); and likewise, what $10^{0.699}$ means.

Write notes on the meaning of powers of ten and how they are combined when multiplying or dividing or taking powers or roots of powers of ten.

Continue to add to these notes during the rest of the section.

Back to the problem, which is to express 5^{100} in scientific notation. You now know that $5 = 10^{0.699}$ (near enough)! Thus:

$$5^{100} \simeq (10^{0.699})^{100} = 10^{(0.699 \times 100)} = 10^{69.9}$$

It just remains to recall that scientific notation requires this last expression to be written as $a \times 10^n$, where a is a number between 1 and 10, and n is a whole number. Now it is easy to express 69.9 as the sum of its decimal part and a whole number: $69.9 = 0.9 + 69$. It follows that:

$$10^{69.9} = 10^{(0.9+69)} = 10^{0.9} \times 10^{69}$$

Using the calculator again, $10^{0.9} = 7.943\,823\,47$ (to eight decimal places). Thus:

$$5^{100} \simeq 8 \times 10^{69}$$

That is certainly a lot more than the population of London! Did you expect there to be quite so many letters in the post at the 100th stage of the chain letter?

Activity 39 Chain gang

(a) How many letters go out at the 8th stage of the chain letter? Give your answer in scientific notation $a \times 10^n$, expressing a to the nearest whole number.

(b) After about how many stages does the number of letters posted exceed the population of London (7×10^6)? (Do not use your calculator!)

Activity 40 Check

What, in the terms of this discussion, is 25 equal to in terms of powers of 10? And 125? How accurate is the calculation?

The conclusion to be drawn is that, in order to express the results of certain calculations involving powers of 5 in scientific notation, it is useful to know the solution to the equation $10^x = 5$. The usefulness of solutions to equations of the form $10^x =$ something, as an aid in calculating, has been known for a long time; indeed, until the invention of the calculator people had to depend almost entirely on tables of such numbers to carry out any complicated numerical calculations at all. These numbers are called *logarithms*, and what you were doing in Activity 36 was calculating $\log 5$—or, to be precise, $\log_{10} 5$ (the logarithm of 5 to base 10).

Historical note

The word *logarithm*, coined in the seventeenth century by the Scottish mathematician John Napier, comes from two Greek words: *logos*, meaning 'ratio', and *arithmos*, meaning 'number'. So logarithms are 'ratio numbers'.

There are some reader articles about the history and development of logarithms: now would be a good time to read them.

Activity 41 Check

Confirm that $\log_{10} 5 = 0.6990$ correct to four decimal places, using your calculator. If by any chance you can lay your hands on a book of four-figure logarithm tables, you might also like to check that the entry for the logarithm of 5 to base 10 is 0.6990.

Finding a logarithm is just 'undoing' the process of exponentiation: the logarithm and the exponential are related to each other in exactly the same way as (for example) multiplying by 2 and dividing by 2; and squaring and taking the square root. That is to say, if you apply one of these operations to some number, and follow this by applying the other operation to the result, then you get back to the number you started from. If you first calculate the logarithm of a number, and then raise 10 to the power of the result, you end up with the number you first thought of. And likewise, if you raise 10 to some power, and then take the logarithm of the result, you end up with the number which was the exponent you began with.

If you have met logarithm tables before, you may be a bit surprised to learn that there is a connection between logarithms and exponentials: there were two types of tables in books of logarithm tables, those for logarithms and those for *antilogarithms*—but no obvious sign of an exponential. But what has just been said about the nature of the relationship between logarithms and exponentials should resolve that puzzle: the function of antilogarithms is to undo logarithms; that is, antilogarithms are exponentials under another name.

In the heyday of logarithms, it was their use as an aid to calculation that was important; and since the first step in carrying out a calculation involved the use of logarithms, it was only to be expected that they got the name and antilogarithms the 'undoing' one. But now it is exponentials that are seen as more basic; logarithms are still important, but their relative importance would be better indicated if they were called (as they might reasonably be if it were not for their history) anti-exponentials.

Activity 42 Graphing log

Graph $y = \log_{10} x$ and $y = 10^x$ together, for both x and y in the range -3 to 3.

Activity 43 Good Queen C

In the Sumwhere National Lottery, Queen Calcula gives the lucky winner one gold piece on the first day of the month, two on the second, four on the third, eight on the fourth, and so on, ending on the last day of a 31-day month. Estimate how many gold pieces she hands out on the last day of the month, given that $\log_{10} 2$ is about 0.3—*without* using your calculator.

The relationship between logarithms and exponentials can be summarized in a couple of formulas:

$$10^{\log_{10} x} = x \qquad \log_{10}(10^x) = x$$

Using these formulas, you can relate the rules for exponents to the rules for calculating with logs. Suppose you have to multiply two numbers together, say x and y. The method for doing this with logarithms is to take the logarithm of each of the numbers, add the two logarithms, and take the antilogarithm of the result. Follow this through now. Add the two logarithms, to get:

$$\log_{10} x + \log_{10} y$$

Taking the antilogarithm is just exponentiating, that is to say, raising 10 to this power. This gives:

$$10^{(\log_{10} x + \log_{10} y)}$$

But the exponential of the sum of two things is the product of the exponentials, so:

$$10^{(\log_{10} x+\log_{10} y)} = 10^{\log_{10} x} \times 10^{\log_{10} y}$$

and since $10^{\log_{10} x} = x$ and $10^{\log_{10} y} = y$, you do indeed end up with $x \times y$.

This is why logarithms were so useful in the days before calculators: they converted the relatively complicated operation of multiplication into the relatively simple operation of addition.

Even though their use as a calculating tool has been superseded by the calculator, logarithms remain important for other reasons. For example, logarithms are used implicitly in the definition of certain special measurement scales. One such scale which you may have come across before is the Richter scale for earthquakes. The Richter scale method of measurement is rather unusual in that it works multiplicatively, not additively as most familiar scales of measurement do. An earthquake which measures 6, for example, on the Richter scale is 10 times more powerful than one which measures 5. In the Richter scale system of measuring the strengths of earthquakes, going up 1 unit on the scale corresponds to multiplying the strength of the earthquake by a factor of 10. In fact, the Richter scale measure of an earthquake is given by the logarithm of its strength.

There is another well known measurement scale which works in a similar way to the Richter scale: the decibel system for measuring the intensity of sound. This is slightly more complicated, because the fundamental unit is actually not the decibel, though this is the more frequently used, but the bel. A sound level of 9 bels is 10 times stronger than one of 8 bels. To give a sound level as so many bels is to give the logarithm of the intensity of the sound. A decibel is simply one tenth of a bel. It is worth bearing this special nature of decibels in mind. If the noise level in your workplace is 'only' going up from 70 decibels (7 bels) to 80 decibels (8 bels), do not be taken in: the noise intensity is actually going to be 10 times greater than it was before.

Before moving to Section 5 where you will be applying what you know about exponentials, you may find it useful to pause and review what you have done and perhaps record your ideas relating to Section 4.

Activity 44 *Identifying learning about exponentials*

Think about what you have done and learned about exponentials. Use the learning outcomes at the end of each section to help you to identify the important aspects relating to exponentials.

Using your notes and activity responses, what could you use to convince someone that you could :

◇ explain what is meant by 'exponential growth';

- ◇ describe the characteristic features of the graph of an exponential function;
- ◇ understand the derivation of the formula for the accumulation and calculate the accumulation, where appropriate, referring to and using the formula;
- ◇ state the relationship between logarithms and exponentials.

Collect your work together and convince yourself (or someone else) that the examples you have chosen do show that you have achieved the outcomes. Now think how you have been learning about exponentials. Include a comment with your work to review what you have learned, and how you have learned it.

Remember to add any further notes on exponentials and logarithms to your Handbook sheet before leaving this section.

Outcomes

After studying this section, you should be able to:

- ◇ state correctly, and use, the rules for exponents (Activity 32);
- ◇ explain (for example, to another student) why they hold (Activities 31, 33);
- ◇ explain what is meant by b^x when x is zero; when it is a negative whole number; when it is a fraction; and carry out calculations with such powers (Activities 34, 35, 38);
- ◇ state the relationship between logarithms and exponentials (Activities 36, 37, 40, 41, 42);
- ◇ explain how logarithms can be used to simplify numerical calculations (Activities 39, 43);
- ◇ select examples of work to show achievements (Activity 44).

5 The joy of exp

Aims This section aims to help you apply what you have learned about exponentials to study some important real-life problems. ◇

As you work through this section, keep your own notes on the concepts introduced for your Handbook.

5.1 Doubling time and half-life

As a result of the algebraic work in Section 4, and the graphical investigations in Section 3, you are now ready to play 'Double Your Money'.

Do you remember investing £200 at an interest rate of 10%, in an account in which the interest is compounded annually? The balance in the account grows according to the exponential function $y = 200(1.1)^x$. (Strictly speaking, this formula gives the correct value for the balance only when x is a whole number; but ignore this detail for the moment; any difficulties it may raise will be sorted out later.) Now since the exponential growth factor, which is 1.1, is bigger than 1, the balance will grow indefinitely large if the money is left undisturbed for long enough. How long will it take for the balance to increase to £400; that is, to reach double the amount initially invested? If the balance eventually gets indefinitely large, then it must certainly reach £400 at some time along the way!

The time it takes for the money in the account to double is a useful measure of how quickly the account balance is growing; it depends on the interest rate, of course, but gives the information about the growth of the balance in a much more accessible way than the interest rate does by itself. This time is called the *doubling time*, naturally enough. A doubling time can be calculated for any population growing exponentially (and here, for once, this really does mean growing, though the word 'population' is again being used in the very general sense introduced in Section 1). Doubling times are often used instead of growth factors to represent the rates of growth of populations.

To find the doubling time for the account, solve the equation:

$$200(1.1)^x = 400$$

or in other words:

$$(1.1)^x = 2$$

This equation shows, straight away, that the doubling time depends only on the interest rate, and not at all on the amount of money invested in the

account: however much you were to invest at a compound interest of 10% per annum, it would take the same time to double.

This equation can be solved graphically, by using the calculator to graph $y = (1.1)^x$ and seeing where the graph crosses the line $y = 2$. But there is another way of doing it, which leads to a general formula for calculating doubling times.

The equation to be solved to calculate the doubling time is similar to the one that was solved in the previous section to calculate 5^{100}. The equation that had to be solved there was $10^x = 5$, and the solution was $\log_{10} 5$. So it should not come as a surprise to learn that the solution to the doubling time problem also involves logarithms. However, they have to be used in a slightly more indirect way.

The logarithm of 1.1 is 0.0414 (to four places of decimals); and the logarithm of 2 is 0.3010. Now remember what this means: it means that $1.1 = 10^{0.0414}$, and $2 = 10^{0.3010}$. The equation to be solved may therefore be written:

$$(10^{0.0414})^x = 10^{0.3010}$$

Using the rule for the exponent of a power we can rewrite this as:

$$10^{(0.0414 \times x)} = 10^{0.3010}$$

One thing may have been apparent to you when you were graphing exponentials: if you raise a given number (other than 1) to two different powers the results you get will be different. So the only way these two powers of 10 can be equal is for the exponents to be equal. Thus, it must be true that:

$$0.0414 \times x = 0.3010$$

so:

$$x = \frac{0.3010}{0.0414} = 7.2705$$

The doubling time is thus about $7\frac{1}{4}$ years. Of course, since the interest is compounded annually, the odd quarter does not really signify; but we can say that the balance will be a little less than double the initial investment after 7 years, and a little more than double it after 8 years. In fact, $(1.1)^7 = 1.95$, while $(1.1)^8 = 2.14$.

All these calculations could have been carried out to a much higher degree of accuracy; but the quoted number of decimal places has been more than adequate for our purposes.

Note that the final expression for x could have been written as:

$$x = \frac{\log_{10} 2}{\log_{10} 1.1}$$

This formula can be entered directly on the calculator to evaluate the doubling time; it is this one which will be generalized below.

Activity 45 *Effects of changing the interest rate*

If the annual interest rate is halved, the growth factor changes from 1.1 to 1.05; if the interest rate is doubled, the growth factor increases to 1.2. Calculate the doubling times in these two cases.

Activity 46 *Inflation and the pound in your pocket*

The same goods cost more each year; this phenomenon is known, of course, as inflation. You know from *Unit 2* that the rate at which the cost of goods is increasing, on average, is measured via the RPI (Retail Price Index); the percentage increase in RPI over a year is called the (annual) rate of inflation. The rate of inflation can vary a lot from year to year, according to the effectiveness of the Government's economic policies, changes in prices of imported raw materials, wage increases, and many other factors. To get an idea of the effects of inflation at different rates, however, it is useful to indulge in the fantasy that the rate will stay constant for a long period of time, and ask how long it will take for the RPI to double at that rate of inflation. The doubling time for the RPI could equally well be described as the time it takes for the pound (£) to halve in value. During the last twenty years (at the time of writing), the rate of inflation in the UK has fluctuated between a low of 2.5% and a high of 25%, roughly speaking. How long would it take the £ to halve (or the RPI to double) in value at each of these extremes?

The method from the savings account problem will now be used to derive a formula for the doubling time for any population growing exponentially, in terms of its exponential growth factor.

Suppose that the growth factor of the population is b, so the function which gives the population size is $y = ab^x$, where a is the initial size of the population.

▶ What is the doubling value, that is, what is the value of x for which the population size is $2a$?

To find it, it is necessary to solve the following equation for x:

$$b^x = 2 \tag{3}$$

Regard b as some fixed known but unspecified number; b must be bigger than 1, or the population will never actually double in size.

The method employed before was to reduce the equation to one relating certain powers of 10, by using logarithms. Now the logarithm of 2 (say) is that power to which 10 must be raised to give 2; this can be expressed in symbols in the form $10^{\log_{10} 2} = 2$. Likewise, $10^{\log_{10} b} = b$; this is just a restatement of what the logarithm of b *is*. So equation (3) can be

expressed as:

$$(10^{\log_{10} b})^x = 10^{\log_{10} 2}$$

So $10^{(x \times \log_{10} b)} = 10^{\log_{10} 2}$

It follows, as before, that x must satisfy this equation:

$$x \times \log_{10} b = \log_{10} 2$$

And therefore:

$$x = \frac{\log_{10} 2}{\log_{10} b}$$

This formula can be used to calculate the doubling time for any population growing exponentially for which the growth factor b is known.

Activity 47 Working backwards

It sometimes happens that you know what the doubling time is for some population growing exponentially, and want to know the exponential growth rate. The relation between the doubling time, d, and the growth rate, b, is:

$$d = \frac{\log_{10} 2}{\log_{10} b}$$

as has just been shown. Can you convert this formula into one which expresses the growth rate b in terms of the doubling time, d? You will need first to express $\log_{10} b$ in terms of d, and then ask yourself how you get b if you know what its logarithm is.

It was shown earlier that the doubling time for a savings account with an annual compound interest rate of 10% is about 7.27 years; so if you invest say £200, the balance in the account will have increased to £400 in 7.27 years. How much longer will it be necessary to wait before the balance doubles again, to £800? Another, similar, question: after one year, the balance will be £220; when will this balance have doubled to £440?

The answer, in each case, is 7.27 years. Once you have calculated the time it takes for your initial investment to double, you know how long it takes for *any* balance to double. The reason for this depends on two facts. First, as was pointed out earlier, the doubling time depends only on the growth rate, not on the amount of money invested initially. Second, there is no essential difference, so far as future balances are concerned, between an account which starts with an investment of £220, and one which has a balance of £220 because it has been going for one year (provided, of course, that the interest rate is the same in each case). The only difference between these two accounts is how the years are numbered.

This is a general truth about doubling times. Suppose that a certain population growing exponentially has a doubling time of d years. Then in

any period of d years, the population will double. In d years from now the population size will be twice what it is now; in $2d$ years from now it will be 4 times what it is now; in $d-1$ years from now it will be twice what it was a year ago; and so on.

This works as follows. The population size is given by $y = ab^x$. The time taken for the population to double from what it was when $x = 0$, namely a, is obtained by solving the equation $2a = ab^d$ for d; so d is determined by the equation $b^d = 2$. This much is known already. Now the population size at any time x is ab^x; the population size d years later is

$$ab^{d+x} = ab^d b^x = 2ab^x,$$

that is, twice what it was to begin with.

It is this universal property of the doubling time which makes it so useful.

The doubling time provides a natural unit of measure for populations growing exponentially. You might care to think about the musical examples from *Unit 9* in these terms: it is helpful to think of the octave as a similar measure—not one of time, but of the 'pitch distance' over which notes double in frequency.

The concept of a doubling time makes sense only for a population which is *growing* exponentially. There is a related idea for an exponentially *decaying* population: its *half-life.* If you are familiar with this expression, you probably met it in the context of radioactivity. It comes up in such statements as 'the half-life of radioactive carbon is approximately 5730 years'.

▶ What does this mean? And why might it matter?

An answer to the second question might be found in the following newspaper report.

> Scientists in Italy and Australia may have cracked the case of the duff Donatello. A terracotta cherub in a church in Florence by the great 15th-century Florentine sculptor contains a giant crack repaired with resin glue. The Italian Ministry of Cultural Heritage sent a sample to the Australian Nuclear Science and Technology Organisation, where the stuff was burned, converted to graphite and bombarded with a heavy metal ion beam to provide a radiocarbon date. And? The results show that the glue was applied between AD 1398 and AD 1439, when Donatello was in business. The cherub may even have cracked in the firing kiln and then been repaired by the great man himself.
>
> (The *Guardian*, 'On Line', 2 March 1995)

Carbon is a chemical element which is found abundantly in the tissues of plants and all living things, and in the atmosphere (in the form of carbon dioxide). Most of the carbon which occurs naturally is non-radioactive. There is a form of carbon which is radioactive, however. The nucleus of an atom of this radioactive carbon—or radiocarbon, for short—is unstable. At any moment it can eject a β-particle, and when it has done so it is no

The scientific detail of this example may be quite unfamiliar to you. Pay particular attention to the use of exponential ideas and the notion of half-life.

longer a carbon atom but an atom of nitrogen. Radiocarbon is chemically the same as ordinary carbon: it forms the same chemical compounds, and behaves in the same way in chemical reactions. If this were not the case, the two substances would not both be called carbon. The difference between the two forms of carbon lies in the composition of their atomic nuclei: the nucleus of an atom of ordinary carbon is stable, and does not disintegrate; but a nuclear reaction can take place in a radiocarbon nucleus which results in its radioactive decay.

Now, what does it mean to say that the half-life of radiocarbon is 5730 years? Suppose you had a lump of radiocarbon, consisting of a large number of atoms. These atoms will spontaneously change into nitrogen atoms, one by one; but it is impossible to tell which particular radiocarbon nucleus will disintegrate when (this is what is meant by describing the decay as 'spontaneous'). However, half of the atoms will have turned into nitrogen in 5730 years; furthermore, half of the remaining atoms of radiocarbon will have turned into nitrogen another 5730 years later; and half of those still remaining will have decayed yet another 5730 years after that. In fact, half of any collection of radiocarbon atoms will decay into nitrogen atoms in 5730 years.

The number of atoms of radiocarbon in a sample decreases exponentially over time. Let y be the number of radiocarbon atoms, t the time measured in half-lives, x the time measured in years and a the initial number of atoms. Then the law giving the number of radiocarbon atoms at any time is:

$$y = a \times \left(\tfrac{1}{2}\right)^t = a \times \left(\tfrac{1}{2}\right)^{x/5730} = a \times \left(\left(\tfrac{1}{2}\right)^{1/5730}\right)^x$$

Now $\left(\tfrac{1}{2}\right)^{1/5730}$ is 0.999 879, to six decimal places, so this can be written as:

$$y = a \times (0.999\,879)^x$$

▶ How is this property of radioactive decay of radiocarbon used to date objects?

The procedure is based on the fact that radiocarbon is present in the carbon dioxide in the atmosphere in small quantities, as a result of the bombardment of the atmosphere by cosmic rays. In fact, the creation of new radiocarbon by cosmic rays pretty well balances the decay of radiocarbon already there, so the proportion of radiocarbon in atmospheric carbon stays constant. When a plant is growing it takes in radiocarbon in the same proportion to ordinary carbon as it exists in the atmosphere. But after the plant dies it no longer absorbs any radiocarbon. The radiocarbon in it undergoes radioactive decay.

The amount of radiocarbon in dead plant material is therefore continually declining. The decay is slow, since the half-life is 5730 years. Nevertheless, over a period of a few hundred years there is a measurable reduction in the radiocarbon level, which can be found by measuring the corresponding reduction in radioactivity. In this way, it is possible to calculate how long

ago a sample of wood or linen or other plant-based material stopped growing, and so date the artefact which the sample comes from.

In the case of the crack in the Donatello cherub, the glue that provided the sample contained resin, which comes from pine trees. Because the half-life of radiocarbon is comparatively long, dating samples like this one which are not terribly old requires great precision. It is possible to estimate the drop in the amount of radiocarbon in the resin by calculating $(0.999\,879)^x$ when x is the age of the sample in years. In this case, the hypothesis being tested is that the crack was repaired in the lifetime of the sculptor, that is, about 600 years ago. Now $(0.999\,879)^{600} \simeq 0.93$. So if the hypothesis is correct, then the amount of radiocarbon in the sample should have dropped by about 7%, compared with the amount there was in it originally. This gives an idea of how small is the effect needing to be measured when radiocarbon dating is used to find the age of something made in historical times.

The first step in the procedure for dating a sample like the glue from the Donatello statue is to extract some of the carbon from it. You know from the newspaper article that this was done in the case of the glue by converting it to graphite. Next, the amount of radioactivity this carbon produces has to be measured in some way. The result can then be compared with how much radioactivity is produced by the same quantity of atmospheric carbon. This shows how much the proportion of radiocarbon in the sample has decreased since its carbon content was fixed. Finally, the exponential decay formula, based on the known half-life of radiocarbon, can be used to calculate the age of the sample.

Activity 48 Carbon dating

A measurement of the radioactivity of a sample of wood from an archeological dig shows that the ratio of radiocarbon to ordinary carbon in the sample is 0.71 times the ratio of radiocarbon to ordinary carbon in the atmosphere. Estimate the date of the site that is being excavated.

Activity 49 Carbon dating: assumptions

The carbon dating method depends for its reliability on an assumption about how the ratio of radiocarbon to ordinary carbon in the atmosphere varies. What assumption is that?

There are many other elements which have isotopes capable of undergoing radioactive decay. In any radioactive decay, an unstable atomic nucleus spontaneously disintegrates by emitting an α- or a β-ray (possibly accompanied by a γ-ray); as a result, the atom of which it is the nucleus changes into an atom of another element. The new atom may again be an unstable isotope.

In general, the number of atoms of a radioactive isotope present will decay exponentially. If you measure time in half-lives, the formula for this exponential decay can be written $y = a \times 2^{-t}$; the minus sign comes from the fact that $\frac{1}{2} = 2^{-1}$. So a half-life is quite similar to a doubling time, except that the half-life tells you *how long ago* there were twice as many atoms of the radioactive isotope in your sample as there are now.

5.2 Interest again

What if the time scale of the exponential system is altered in some way? One interesting case to consider is changing the frequency with which interest is compounded.

Example 7 The APR

APR stands for *annual percentage rate*. If you have a credit card, the interest charged on any debts you owe to the credit card company will be quoted as an APR. But statements are presented monthly; and if you fail to pay off your debt at the end of one month, then you will be charged interest on the sum outstanding on the next statement. So interest is charged on a monthly basis. The idea of the APR is to present the interest rate in a form which enables you to make a sensible comparison between using a credit card as a form of loan, and other forms of loan, such as bank loans, for which interest is both quoted and charged on an annual basis.

The APR is calculated on the following entirely hypothetical assumptions. Suppose someone has a credit card debt of £100 at the beginning of the year, and that he makes no transactions and pays nothing off during the year. Interest will be charged at a fixed monthly rate; and each month the interest owed will be added to the sum outstanding. The interest therefore compounds monthly. The *total* amount of interest owing at the end of the year is the APR for that account.

▶ What is the APR if the monthly interest rate is 2%? What is the monthly interest rate if the APR is 24?

If the monthly interest rate is 2%, and you owe £100 initially, then after one month you owe £102. After two months you owe this amount plus an additional 2%, that is, $£(1.02)^2 \times 100$. This process continues on the familiar exponential pattern; after one year you owe $£(1.02)^{12} \times 100$. Now $(1.02)^{12} \simeq 1.268\,241\,795$, so the sum you owe at the end of the year is £126.82 (to the nearest penny). The total amount of interest owing is £26.82, and the APR is therefore 26.82. (It is not necessary to use a % sign, since the fact that it is a percentage is already part of the name.)

Note that the calculation of the APR involves exponentiation; the APR is not just the monthly rate times the number of months in a year, and in fact is larger than what you would get by calculating that product. This indeed is one of the reasons why every credit company has to quote the interest rate on every type of loan in the form of an APR; people are easily misled by small rates compounded over short times into thinking that they are getting a bargain when on an annual basis they are paying much more.

The converse question is: when the APR is 24, what is the monthly interest rate? When the APR is 24, the total interest on £100 owing at the end of the year is £24; the total sum owing (original debt plus interest) is therefore £124. Thus, if $x\%$ is the unknown monthly interest rate, then $(1 + x/100)^{12} \times 100 = 124$, or $(1 + x/100)^{12} = 1.24$. The first step towards solving this equation is to find the 12th root of 1.24: $\sqrt[12]{1.24} \simeq 1.018\,087\,582$. Thus, $1 + x/100 = 1.018\,087\,582$, so $x = 100 \times 0.018\,087\,582 = 1.808\,758\,2$. The monthly interest rate corresponding to an APR of 24 is therefore 1.81% (to two decimal places).

Activity 50 APR revisited

Find the APR for a monthly interest rate of 1%, and the monthly interest rate for an APR of 12.

Activity 51 More frequent interest

The concept, and definition, of the APR are not just restricted to monthly payments, though that is the most common case. Find the APR for an interest rate of 1% compounded twice a month (that is, 24 times a year).

Activity 52 More on interest

You need a loan of £1000, and want to pay as little interest as possible. You are offered a choice of three fixed-interest loans:

(a) an interest rate of 6.75% compounded annually;

(b) an interest rate of 3.25% compounded every six months;

(c) an interest rate of 0.5% compounded monthly.

Which option should you choose?

Activity 53 *Interest again*

You are offered the choice of two savings accounts, paying fixed interest for one year:

(a) an interest rate of 0.03%, compounded daily;

(b) an interest rate of 0.2%, compounded weekly.

Which will give you the best return on your savings? (Assume 1 year = 52 weeks = 364 days.)

When reading an advertisement offering a loan at a certain APR, with interest to be repaid monthly, it is natural to make a quick estimate of the monthly interest rate by dividing the APR by 12. But the examples above show that this leads to an overestimate of the monthly interest rate. As an apparently very simple further example, consider the case where the APR is 100, so that the sum owing doubles in a year, the sum owed originally is £1, and the interest is compounded just twice. If the interest rate is taken to be half of the APR, then the total owing at the end of the year appears to be $£(1.5)^2 = £2.25$, compared with the sum of £2 according to the correct calculation.

The effect of the approximation (of simply dividing the APR by the number of times the interest is compounded) is to produce an overestimate of the total sum owing of £0.25 in every £, in this case. If the number of times the interest is compounded is increased, say to 4, then the error in the approximation gets worse: the quarterly interest rate is taken to be 25%, and the total owing appears to be $£(1.25)^4 = £2.44$. Increasing the number of times the interest is compounded increases the error again. It is worth finding out just how bad the error gets: you are asked to investigate this in the next activity. You may find the result surprising; it is certainly true to say that it has important consequences.

Activity 54 *A table*

Use your calculator to complete the following table for an APR of 100.

Number of times per year interest is compounded	Total sum owing at end of one year in £
1	2
2	2.25
4	2.44
12	2.61
24	2.66
52	
365	
500	
1000	
10000	

In fact, what you have been computing is the expression

$$\left(1+\frac{1}{n}\right)^n$$

for increasing values of n, the number in the left-hand column. As n gets larger and larger, the number in the right-hand column gets closer and closer to the value

$$2.718\,281\,828\,5\ldots$$

This is a very important constant. It is denoted by the letter e, and is used in very many ways throughout mathematics. It is rather like π in some respects: in particular, the decimal expansion of e, like that of π, does not terminate nor recur. This means that e, like π, cannot be represented as a fraction, since the decimal expansion of any fraction either terminates (like $\frac{1}{2}$ and $\frac{1}{4}$) or is repeating (like $\frac{1}{3}$ and $\frac{1}{7}$). Numbers like this are called *irrational*. (In case you are moved to protest that $\pi = \frac{22}{7}$, remember $\frac{22}{7}$ is only *an approximation* to the value of π, though quite a good one.)

A rational number is one that can be expressed as a fraction, as a *ratio* of two whole numbers. An irrational number, such as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, is one that cannot: 'ir-rational' means 'without ratio', but the association with other mental processes and 'ratio-nality' itself are strong and significant.

5.3 Continuous exponential growth

So far, you have been dealing with processes which go in distinct stages: one day at a time, in the case of Queen Calcula's Bounty; one year at a time, in the case of trees; one generation at a time, in the case of human ancestors. But some similar processes vary continuously. Both the density and the pressure of the atmosphere decrease with increasing height above sea level (this explains why mountaineers sometimes carry oxygen when climbing Mount Everest, and why the cabins of aircraft are pressurized); as it happens, both density and pressure are exponentially decreasing functions of height. But in neither case could you sensibly use the idea of 'a generation' in the analysis of how the quantity varies.

These examples are similar to those discussed so far in the unit, in that they involve exponential functions; but they are different in that both the independent variable (the height) and the dependent variable (the density or the pressure) vary continuously.

In the case of Queen Calcula's Bounty, both variables (the day of the month, and the number of gold pieces) can take values which are whole numbers only. They cannot be said to vary continuously, therefore; rather, they change by jumping discontinuously from one whole number to another. The word used to describe models which show this jumpiness is 'discrete': Queen Calcula's Bounty (like most of the models discussed so far) is a discrete exponential model, whereas the exponential model of the decrease of atmospheric pressure with height is a continuous one.

Recall from *Unit 4* the distinction between discrete and continuous variables.

Even though the models in this unit are discrete, they have been analysed in some ways as though they were continuous. The functions of the form $y = ab^x$ are certainly continuous, as you know from examining typical graphs. The discussion of fractional powers showed how these graphs contrive to be continuous curves. You have seen how to make the transition

from the continuous function to the discrete situation in the discussion of problems involving money: simply use the continuous exponential model for the calculation, but round to the nearest penny in giving the answer.

There is a kind of intermediate state between the out-and-out discreteness of the problem of Queen Calcula's Bounty, and the out-and-out continuity of atmospheric pressure. You met this intermediate state when discussing radioactive decay. Notionally, the dependent variable in this case is a number (of atoms); it should therefore be regarded as taking discrete values. Moreover, the events which lead to the exponential decrease in the number of atoms (of the unstable isotope) are the discrete events of the radioactive decays of individual atoms. But any practical case means dealing with such enormous numbers of atoms that individuals count for scarcely anything. Furthermore, the times at which the decays take place are completely unpredictable, and quite unlike the regular occurrences of the increases in Queen Calcula's Bounty. In a case like this, there is really no alternative to treating the process as though it were a continuous one. Actually, you have been treating similar situations as though they were continuous all along. Consider, for example, Figure 7.

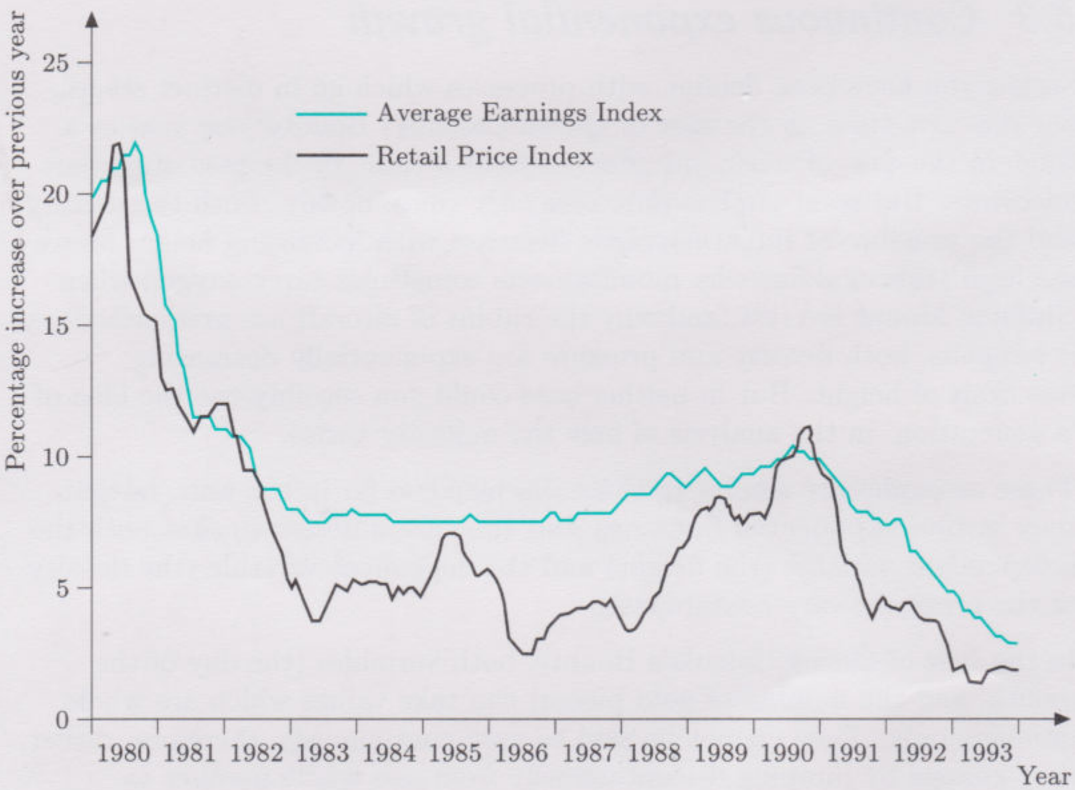


Figure 7 Changes in the AEI and the RPI: 1980–1993

Figure 7 shows changes in the Average Earnings Index (AEI) and the Retail Prices Index (RPI) over the period 1980–1993. Both graphs are shown as continuous curves. The horizontal axis represents time, with the years marked on it. The idea of a continuous curve is that if you look at it under a magnifying glass, however powerful, it remains continuous ('joined up'). In other words, you can enlarge the scale, so that, for example, the year 1984 can be made to fill the whole of the horizontal axis on your page, as shown in Figure 8.

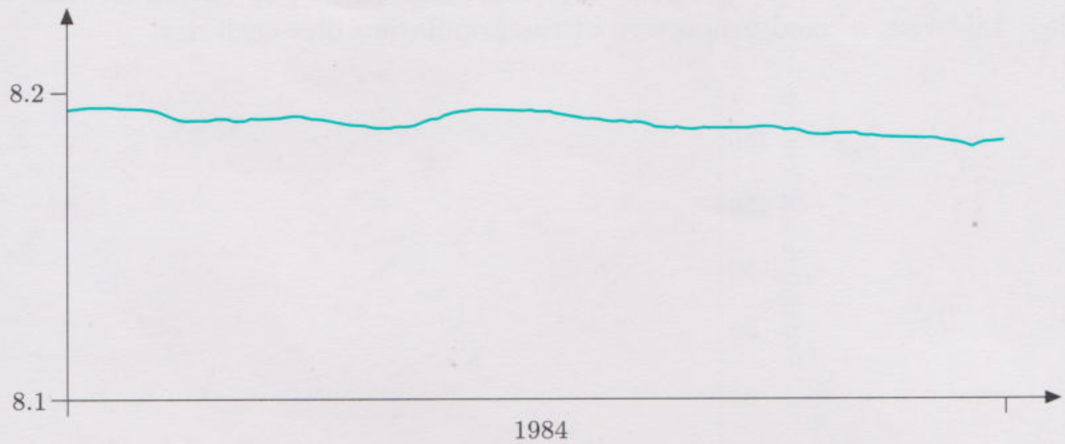


Figure 8 Change in AEI: 1984

In principle, you can go on doing this, enlarging the graph more and more, so that you can examine the graph for just one month in 1984, then just one week, then a day, an hour, a minute, a second, and so on. But obviously this does not make sense! The AEI is not worked out over such short intervals of time—monthly is the best that can be done. Indeed, in any real-life situation time is always measured in *discrete* amounts of some finite length—for example, hundredths-of-a-second when timing sporting events; or three months for bills like gas or electricity.

But even if the continuous model is, strictly speaking, a fiction, it is a very useful one.

5.4 Populations of people

As part of your work in this subsection, you will have to use the calculator to find the best exponential fit to some given data. The technique for doing this is explained in the *Calculator Book*.

Now study Section 12.4 of Chapter 12 in the Calculator Book

The growth of a real population (that is, a population of people) is a good example of a process that has to be studied via a continuous model, even though individual persons cannot be subdivided.

Consider the population of some country: the USA, for example. It keeps on growing; and as you can see from Figure 9 it does so (more or less) exponentially.

The reason it does grow exponentially is that, in any fixed small period of time (say a day, but it could be an hour, or a week—choose whichever you fancy), the increase in the population is proportional to the size of the population. How many babies are born depends, after all, on how many women there are to have them; the bigger the size of the population, then (other things being equal) the more babies will be born; and one way of modelling the growth of the population is to assume that the number of

babies born each day is proportional to the size of the population on that day. Likewise, a fixed proportion of the population dies each day.

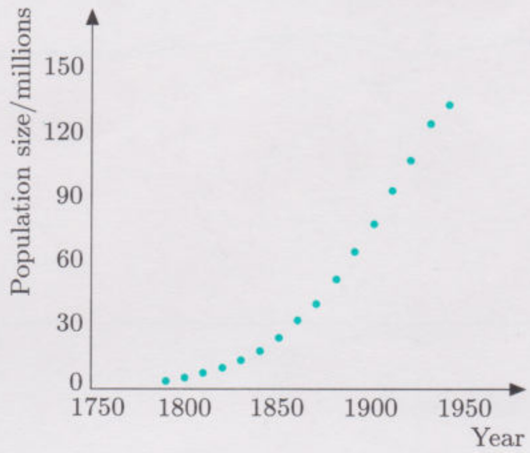


Figure 9 Population of the United States, 1790–1950

Express the way the population changes as follows:

$$\text{population at end of day} = \text{population at beginning of day} + \text{births} - \text{deaths}$$

If $P(n)$ is the size of the population on the nth day, B is the birth rate (the number of live births per head of the population per day) and D is the death rate (the number of deaths per head of the population per day), then

$$\begin{aligned} P(n+1) &= P(n) + BP(n) - DP(n) \\ &= (1 + B - D)P(n) \end{aligned}$$

This is the familiar formula for exponential growth (assuming, of course, that $B > D$).

This explains why you might expect the population to grow exponentially; but it does not provide a very good way of measuring the exponential growth factor, because birth and death rates vary a lot among different sectors of any large, heterogeneous population. It is better to gather data on the population size, and fit an exponential function to them. Table 2 gives data for the world's population taken from two UN publications.

Table 2

Year	Population
1850	1.094×10^9
1900	1.500×10^9
1925	1.907×10^9
1930	2.070×10^9
1950	2.500×10^9
1960	3.019×10^9
1990	5.292×10^9

(The data for 1930, 1960 and 1990 come from the *United Nations Demographic Yearbook 1990*; the remaining data from *The Future Growth of World Population* (UN, 1958).)

Activity 55 Investigating these data

Use your calculator to make a scatterplot of these data; use ten years as your unit of time.

Find the values of a and b for which the graph of $y = ab^x$ best fits the data. With these values of a and b, superimpose the graph of $y = ab^x$ on your scatterplot, so that you can see how well the exponential model fits the data.

Activity 56 Interpret your results

It should be clear to you that in the more recent years for which data are available, the population has been growing more quickly than the historic trend would have suggested. Can you offer any explanations for this discrepancy?

One of the best ways of understanding the consequences of continuing growth in the world's population is to work out how long it will take for the population to double; in other words, to calculate its doubling time. This is a statistic which is often quoted in discussions of world population growth.

Activity 57 Population doubling time

Using the best exponential fit to the data from the previous activity, find the corresponding doubling time.

In the thirty years from 1960 to 1990, the population increased more than the model based on the historical data would predict. In fact, the ratio of the 1990 population to the 1960 population is $5.292 \div 3.019 = 1.753$. This is for a period of thirty years, of course, so cannot be compared directly with the growth factor determined in the activity.

Activity 58 Population doubling time—a second look

Find the growth factor which gives a proportional increase in the population of 1.753 in 30 years. Take 10 years for the unit of time, as before. Calculate the doubling time for a population with this growth factor.

The population of the world is actually growing more quickly even than an exponential model would predict. You can get an idea of how its rate of growth is increasing by calculating the 10-yearly growth factor for all the periods for which there are data in Table 2. A scatterplot of these growth factors against time shows a lot of variation, but an increasing trend; linearly extrapolating the factor forward to the year 2000 and calculating the corresponding doubling time gives a value of about 32 years. That is to say, if this analysis is at all reliable, and the trend continues, in the year 2000 the population will be growing so fast that it will have doubled before 2032.

Remember to check that you have good notes on doubling times and half-lives for exponential growth and decay.

Outcomes

After studying this section, you should be able to:

◇ define the terms 'doubling time' and 'half-life';

◇ calculate the doubling time or half-life (as appropriate) for an exponentially growing population, given its growth factor (Activities 45, 46, 47, 48);

◇ explain what an APR is, and calculate the APR (for example) for a loan for which interest is charged at some period other than a year (Activities 50, 51, 52, 53, 54);

◇ explain how changes in the growth factor affect the growth of a population, such as the world population (of people), which is growing in an approximately exponential manner (Activities 57 and 58);

◇ fit an exponential regression model to given data (Activity 55).

6 The *exponential*

Aims This section aims to introduce you to a very special exponential function. ◇

The facts that $b^{cx} = (b^c)^x$, and that given b you can always find another number c so that b^c has some pre-assigned value, mean that you can obtain all exponential functions from just one of them, which might reasonably be called *the* exponential function. This was important before calculators came along: then people had to look up the values of 2^x, 10^x, b^x, ... in tables, and you can see how complex it would have been if you had to have a different set of tables for all of these different but related functions. It remains important for different reasons, though it is not so easy to explain why: this will become clearer if you take further courses in mathematics.

There would appear to be two obvious candidates for *the* exponential. The first is $y = 10^x$. It is an obvious candidate because most peoples now count in powers of 10, and scientific notation is already halfway to being exponential based on 10. The other is $y = 2^x$, which would take advantage of the usefulness of doubling time and half-life, both well established concepts.

As it happens, neither of these is taken as *the* exponential. *The* exponential is $y = e^x$ where e is the number that you calculated in Activity 54. There are some very good, but not obvious, reasons for this rather peculiar looking choice, which we shall try to explain towards the end of this section.

In the next subsection, you return to the calculator to investigate *the* exponential function.

6.1 The *exponential graph*

It is worth checking that the graph of $y = e^x$ is similar to the graphs of other exponentials which you investigated earlier.

Now study Section 12.2 of Chapter 12 of the Calculator Book

6.2 *Why e?*

The main difference among graphs of $y = b^x$ for different values of b is in their steepness: the larger b is, the steeper the graph is. All of the graphs go through the point $(0, 1)$. When $b = 1$, the graph is a horizontal straight line: it is dead flat; its gradient is zero. When $b = 10$, say, the graph is very steep as it passes through that point. It is not a straight line, of course, so properly speaking it does not make sense to talk about its

gradient in the same way; nevertheless, it is clear that if it did, the gradient at $(0, 1)$ would be large. As b increases from 1, the gradient of the graph of $y = b^x$ at the point $(0, 1)$ gets larger and larger. When $b = 1$ the gradient is 0; when $b = 10$ the gradient is certainly bigger than 1. There is some value of b between 1 and 10, therefore, for which the gradient of the corresponding graph at the point $(0, 1)$ is exactly 1.

If you wanted to pick out one particular exponential function as special, in a way which does not depend on accidents like having ten digits on our hands, but is intrinsic to the very nature of exponential functions themselves, the one whose gradient at $(0, 1)$ is 1 looks like a good one to choose. This is precisely what is done: the value of b for which the corresponding exponential has gradient 1 at $(0, 1)$ is the number $e \simeq 2.718\ldots$, and this is why $y = e^x$ is known as *the* exponential function.

This special property of $y = e^x$ can be easily seen using the calculator.

Activity 59 The gradient of the exponential

Graph the function $y = e^x$, in a small region around the point $(0, 1)$, making that point the centre of your screen, and setting up the viewing window so that the units on the x and y axes are of equal length on the screen (so that lines of gradient 1 appear at 45° to the axes). Now zoom in two or three times, keeping $(0, 1)$ at the centre of the screen. At each zoom, the graph appears more nearly like a piece of a straight line. This is a general feature of zooming in on graphs. The particular point of interest in this case is that the line has gradient 1. It should certainly look as though it does, to the naked eye. But this can also be checked, using the trace facility.

When you have zoomed in sufficiently often that the graph looks good and straight, trace along the graph, noting the x and y coordinates as you proceed. You should find that they both change by pretty well the same small amount at each step. This will be most obvious as you trace to the right, since both coordinates increase. When you trace to the left of the point $(0, 1)$, the coordinates decrease in step; but you have to be able to do subtraction in your head to realize this! Here are some sample values obtained by this process.

x	y
0	1
0.00136	1.0014
0.00272	1.0027
0.00408	1.0041

x	y
0	1
−0.0014	0.99864
−0.0027	0.99729
−0.0041	0.99593

The conclusion of this activity is, firstly, that it does make sense to talk about the gradient of the graph of $y = e^x$ at the point $(0, 1)$, since repeated zooming quickly reduces the graph to what is effectively a straight line; secondly, that this gradient is 1. It is this latter property that defines *the* exponential function.

Just as you can work out exponentials to any base b, $y = b^x$, you can work out logarithms to any base b, $y = \log_b x$. And when the two values of the base are equal, the two particular functions are inverse to each other. So $y = \log_{10} x$ and $y = 10^x$ are inverse to each other. Similarly, $y = \log_2 x$ and $y = 2^x$ are inverse to each other.

Corresponding to *the* exponential, $y = e^x$, there is a particular logarithm function, namely logarithms to the base e, $y = \log_e x$. The function $y = \log_e x$ is often denoted by $\ln x$, which stands for 'natural logarithm' (or more likely '*logarithmus naturalis*'). Thus, $y = \ln x$ is the function inverse to $y = e^x$; this means that $e^{\ln x} = x$ and $\ln(e^x) = x$. You will usually find that e^x and ln share a key on a calculator, for this reason.

The exponential and logarithm functions are the preferred choices when using exponentials for theoretical purposes in mathematics and science, and you are bound to come across them again if you take your mathematics further than this course.

Activity 60 *Learning and reflecting about exponentials*

Use your notes to comment on:

- things that were useful in helping to learn and understand about exponentials (for example, seeing, speaking and recording; or explaining to a friend);
- any aspect you found difficult and what you did about it;
- a justification of the activities/work you selected to complete;
- a self-evaluation of how well you have learned about exponentials.

Outcomes

After studying this section, you should be able to:

- describe the graph of *the* exponential function $y = e^x$;
- give at least one reason why this particular exponential is singled out for special consideration in mathematics (Activity 59);
- actively and regularly choose activities to improve learning.

Unit summary and outcomes

Just as *Unit 10* looked at the family of linear functions and *Unit 11* the family of quadratic functions, this unit has focused on a third family, that of exponential functions $y = ab^x$ (and their related inverse functions the *logarithms*). Applications included financial, biological and chemical examples and later ones included discussions of the important idea of doubling time and half-life.

Outcomes

You should now be able to:

- ◇ explain what is meant by 'exponential growth', recognize examples of exponential growth, and distinguish exponential growth from other types of change, such as linear or quadratic change;
- ◇ write down the general formula for the population size of a population growing exponentially;
- ◇ interpret descriptions of exponential models;
- ◇ decide, in a given situation involving exponential growth, when it is appropriate to calculate the accumulation over several generations rather than just the population size for any given stage;
- ◇ understand the derivation of the formula for the accumulation;
- ◇ calculate the accumulation, where appropriate, referring to and using the formula;
- ◇ describe the characteristic features of the graph of an exponential function;
- ◇ recognize exponential graphs, and in particular distinguish them from graphs of other classes of functions such as linear functions and quadratics;
- ◇ describe how the values of the constants a and b affect the shape and position of the graph of $y = ab^x$;
- ◇ describe the graph of the modified exponential function $y = ab^x + c$;
- ◇ draw conclusions about the long-term behaviour of the accumulation of a population when the growth factor of the population is less than 1;
- ◇ state correctly, and use, the rules for exponents;

- ◇ explain (for example, to another student) why they hold;
- ◇ explain what is meant by b^x when x is zero; when it is a negative whole number; when it is a fraction; and carry out calculations with such powers;
- ◇ state the relationship between logarithms and exponentials;
- ◇ explain how logarithms can be used to simplify numerical calculations;
- ◇ define the terms 'doubling time' and 'half-life';
- ◇ calculate the doubling time or half-life (as appropriate) for an exponentially growing population, given its growth factor;
- ◇ explain what an APR is, and calculate the APR (for example) for a loan for which interest is charged at some period other than a year;
- ◇ fit an exponential regression model to given data;
- ◇ explain how changes in the growth factor affect the growth of a population, such as the world population (of people), which is growing in an approximately exponential manner;
- ◇ describe the graph of *the* exponential function $y = e^x$;
- ◇ give at least one reason why this particular exponential is singled out for special consideration in mathematics.

Comments on Activities

Activity 1

Making ideas your own is an important part of learning and so it seems reasonable to pay some attention to the transitions from seeing something, to saying what you see, to recording it. In particular, it is often helpful to use pictures and bits of sentences before trying to get complete thoughts. You may have found that it is much easier if you can be clear about what you see (think how you have been recording entries for your Handbook), so it is worth spending time telling and re-telling someone (or the dog!) how you see what is going on. The attempt to talk about some idea, and then write it down, is an attempt to capture or crystallize it in some form so that it gets out of your brain. Then you can look at it afresh.

Activity 2

There are no comments on this activity.

Activity 3

(a) Exponential; the bacterial cells; a minute; 2.

(b) Not exponential.

(c) Exponential; the things coming from St. Ives; a line in the poem; 7.

(d) Not exponential.

Activity 4

The following example comes from the Vice-Chancellor's Introduction to the Open University's Annual Review for 1995: '1994 was also the year when the term 'Information Superhighway' came into common parlance and the use of the Internet increased exponentially. The Open University is committed to maintaining its world leadership in the large scale application of technology to higher education.'

Activity 5

5^3; 5^{17}; 5^n.

Activity 6

$P = 5^{10} = 9\,765\,625$.

Activity 7

(a) For simple interest the same interest is added each year, so the amount of interest added per year is £21.70. This is the interest earned by £500, so the rate is 4.34%.

(b) For the compound interest case, you have to find a number which gives 608.5 $(108.5 + 500 = 608.5)$ when it is raised to the fifth power and the result then multiplied by 500; that is, a number whose fifth power is $608.5 \div 500 = 1.217$. Unless you already know a better way, at this stage there is no alternative to trial and error. You know that the interest rate for compound interest will be less than that for simple interest. So try 4%, with growth factor 1.04. $(1.04)^5 \simeq 1.216\,652\,902$, which is near enough; a lucky guess!

Activity 8

$£p \times (1.1)^n$

Activity 9

If she invests $£p$, then after eighteen years she will have $£p \times (1.05)^{18}$. She wants to choose p so that this comes to £1 000 000. Now, $(1.05)^{18} = 2.406\,619\,234$, so $p = 1\,000\,000 \div 2.406\,619\,234 = 415\,520.65$, and so the sum required is £415 520.65 to the nearest penny.

Activity 10

(a) There will be 17×3^n simplified greenfly n days later.

(b) None, you have to do the calculation only once.

Activity 11

(a) $P(n) = 3 \times \left(\frac{4}{3}\right)^n$

(b) When the sides of the initial triangle are all $\frac{1}{3}$, its perimeter is 1, so $P(n) = \left(\frac{4}{3}\right)^n$. If the perimeter of the initial triangle is 5, then $P(n) = 5 \times \left(\frac{4}{3}\right)^n$

(c) There is no comment for this part.

Activity 12

(a) The whole population is double the size of the female population (you are told that half the young are females; in the absence of any other information, assume only that the survival rates for males and females are the same). The growth rate is the same for the whole population.

(b) The model population sizes in successive years are:

$$\begin{aligned} 1000 \times 2.06 &= 2060 \\ 1000 \times (2.06)^2 &= 4244 \\ 1000 \times (2.06)^3 &= 8742 \\ 1000 \times (2.06)^4 &= 18\,008 \end{aligned}$$

These figures are in reasonable agreement with the data shown on the graph—a little low, but then it really is rather difficult to get a good estimate of the population size in 1960.

(c) k is the proportion of adult females which survive from one year to the next, that is, the adult survival rate; this is given as 86%, but you need it as a rate, not a percentage, and 86% converts to 0.86. Likewise, l is the proportion of young birds surviving to the next year, or the juvenile survival rate, which is given as 60%, which converts to 0.60. m is the number of young, and you are told that 'each pair has the potential to rear four young'; it is clearly assumed that they all realize that potential!

Substituting the values 0.86, 0.60 and 4 for k, l and m, respectively, into the equation $b = k + \frac{1}{2}ml$ gives $b = 2.06$.

(d) The population size will increase sharply through the breeding season, as young are produced in great numbers; but it will then gradually decline until the beginning of the next breeding season as old age and winter take their toll. It is therefore important to specify carefully when the population is measured, in order to compare like with like from one year to the next, and to ensure that what is predicted by the model corresponds to what is counted by the ornithologists in the field.

(e) $kP(n)$ is the 'Number of adult females in breeding population in year n $\times$ the proportion of adults surviving to year $(n+1)$' (but in the reverse order). So far as the other term is concerned, the 'number of young females produced in year n' is obtained by taking the number of breeding pairs (assuming that a pair consists of one of each gender), which is the same as the number of breeding females and multiplying by the average number of female young per breeding pair, which is half the average number of young per breeding pair. So this gives $\frac{1}{2}mP(n)$. Finally, this must be multiplied by the survival rate, which is l.

(f) As a species new to the British Isles, it probably found a niche for itself: that is, it found little competition for its food and few predators. The population cannot continue to grow exponentially indefinitely into the future, because if it did it would get arbitrarily large. Eventually growth must slow down, as competition for food and living space take over, and the population size will level off.

Activity 13

$S(n) = 2^n - 1$, by more or less the same argument.

Activity 14

$2^{28} - 1 = 268\,435\,455$ in ordinary years;
$2^{29} - 1 = 536\,870\,911$ in leap years.

Activity 15

$1 + 7 + 7^2 + 7^3 + 7^4 = (7^5 - 1)/(7 - 1) = 16\,806/6 = 2801$ (though there is some doubt, given the wording of the poem, about whether you should count the man).

Activity 16

When forming the cumulative sum of the population sizes generation by generation, note that each term in the sum is a times the corresponding term in the sum already treated; the total simply gets multiplied by a, and so:

$$S(n) = a\left(\frac{b^n - 1}{b - 1}\right)$$

Activity 17

$1000 \times (1.07)^5 = 1402.55$, so without withdrawals you have £1402.55. With withdrawals, this has to be reduced by

$$100 \times \left(\frac{(1.07)^5 - 1}{1.07 - 1}\right) = 575.07,$$

so you are left with £827.48.

Activity 18

One way of approaching the problem is to treat the history of the half ration of water already in the soil separately from the rest of the process. It may help you to imagine this if you think of the initial half ration as being physically distinguishable from the water that is delivered by irrigation (perhaps, being rainwater, it has a slightly different composition from the irrigation water). All that happens to the initial water is that half of however much there is of it remaining is lost by evaporation each day; so it just decreases exponentially with factor $\frac{1}{2}$. Since there is a half ration to begin with, the amount remaining on the nth day is $\frac{1}{2} \times \left(\frac{1}{2}\right)^n$. The amount of irrigation water in the soil is given by the formula developed in the text. The total amount of water is just the sum of the two parts, and is therefore given by:

$$\begin{aligned}&\tfrac{1}{2} \times \left(\tfrac{1}{2}\right)^n + \left(2 - \left(\tfrac{1}{2}\right)^{n-1}\right)\\ &= 2 + \tfrac{1}{4} \times \left(\tfrac{1}{2}\right)^{n-1} - \left(\tfrac{1}{2}\right)^{n-1}\\ &= 2 - \tfrac{3}{4} \times \left(\tfrac{1}{2}\right)^{n-1}\end{aligned}$$

Alternatively, carry out the calculation as in the text. If $w(0) = \frac{1}{2}$, then:

$$\begin{aligned}w(1) &= \tfrac{1}{4} + 1 = \tfrac{5}{4}\\ w(2) &= \tfrac{1}{2} \times \tfrac{5}{4} + 1\\ w(3) &= \tfrac{1}{2} \times \left(\tfrac{1}{2} \times \tfrac{5}{4} + 1\right) + 1 = 1 + \tfrac{1}{2} + \tfrac{5}{4} \times \left(\tfrac{1}{2}\right)^2\\ w(4) &= 1 + \tfrac{1}{2} + \left(\tfrac{1}{2}\right)^2 + \tfrac{5}{4} \times \left(\tfrac{1}{2}\right)^3\end{aligned}$$

and so on. The extra half ration of water at the start is attached to the term with the highest power of $\frac{1}{2}$ in it; this gives the clue to the formula for the nth day, which is

$$w(n) = 1 + \tfrac{1}{2} + \left(\tfrac{1}{2}\right)^2 + \ldots + \left(\tfrac{1}{2}\right)^{n-2} + \tfrac{5}{4} \times \left(\tfrac{1}{2}\right)^{n-1}$$

You can simplify this by using the sum formula for the terms up to $\left(\frac{1}{2}\right)^{n-2}$, which gives:

$$\begin{aligned}w(n) &= \frac{1 - \left(\frac{1}{2}\right)^{n-1}}{1 - \frac{1}{2}} + \tfrac{5}{4} \times \left(\tfrac{1}{2}\right)^{n-1}\\ &= 2 - 2 \times \left(\tfrac{1}{2}\right)^{n-1} + \tfrac{5}{4} \times \left(\tfrac{1}{2}\right)^{n-1}\\ &= 2 - \tfrac{3}{4} \times \left(\tfrac{1}{2}\right)^{n-1}\end{aligned}$$

as before.

Activity 19

(a) $y = 3 \times \left(\frac{4}{3}\right)^x$; exponential; $a = 3$, $b = \frac{4}{3}$.

(b) $y = l \times \left(\frac{4}{3}\right)^x$; exponential; $a = l$, $b = \frac{4}{3}$. (Note that l is a parameter not a variable.)

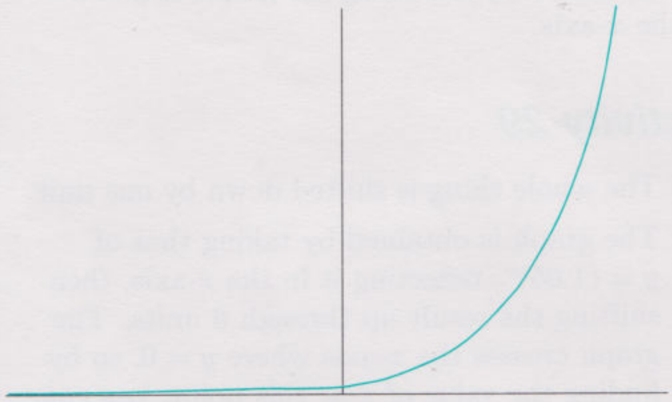
(c) $y = 5^x$; exponential; $a = 1$; $b = 5$.

(d) $y = 2^x - 1$; variant; $a = 1$, $b = 2$, $c = -1$.

(e) $y = 6000 - 1000 \times (1.05)^x$; variant; $a = -1000$; $b = 1.05$; $c = 6000$.

Activity 20

Your graph should look like the following.



The graph of $y = 3^x$ is always going uphill to the right, more and more steeply. A straight line (with positive slope) always goes uphill to the right, but with no change of steepness. A parabola (the right way up) goes uphill as you move to the right of its vertex; but it also goes uphill as you move to the left of its vertex.

Activity 21

(a) $y = 1$ when $x = 0$; $y = 3$ when $x = 1$; $y = \frac{1}{3}$ when $x = -1$.

(b) 3^x gets rapidly larger and larger as x increases from 0; it gets closer and closer to 0 as x decreases from 0.

Activity 22

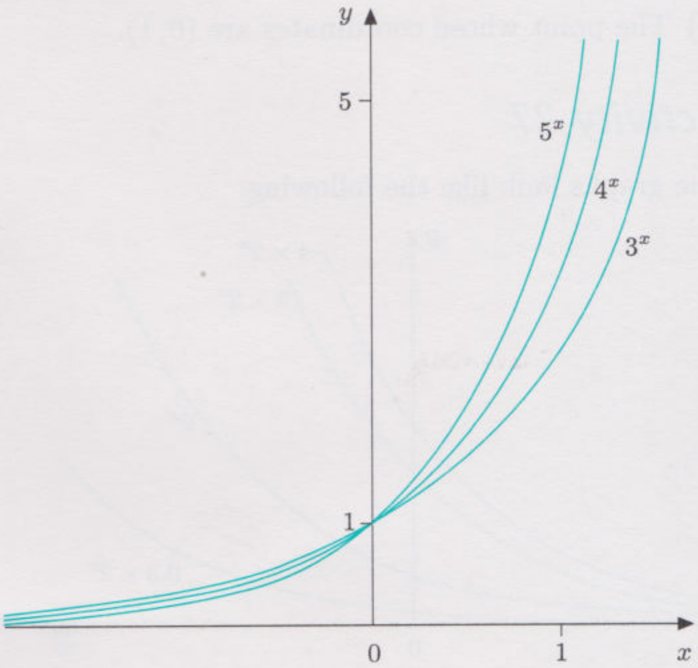
In addition to what has been said in the comments on Activity 21, note the following differences: the straight line goes below the x-axis, the exponential never does; the parabola is symmetric, the exponential is not.

Activity 23

As x tends to $+\infty$, $2x + 1$ tends to $+\infty$; as x tends to $-\infty$, $2x + 1$ tends to $-\infty$. As x tends to $+\infty$, $2x^2 + 1$ tends to $+\infty$; as x tends to $-\infty$, $2x^2 + 1$ tends to $+\infty$.

Activity 24

Your graphs should look something like the following.



(a) $y = 1$ when $x = 0$ in all cases; $y = 3$, 4 and 5, respectively when $x = 1$.

(b) Each goes uphill to the right, more and more steeply. In fact, they all have much the same shape. The graph of $y = 4^x$ lies above $y = 3^x$ to the right of the y-axis, but below it to the left; the graphs cross on the y-axis (where $x = 0$ and $y = 1$). Similarly for $y = 5^x$. The graph of $y = 6^x$ lies above the other three to the right of the y-axis, but below them to the left; $y = 2^x$ lies below the other three to the right of the y-axis, but above them to the left; $y = (3.5)^x$ lies between $y = 3^x$ and $y = 4^x$.

(c) All graphs of $y = b^x$ have the same general shape, for $b > 1$; if $B > b$, $y = B^x$ lies above $y = b^x$ to the right of the y-axis, but below it to the left; the graphs cross on the y-axis, at the point whose coordinates are $(0, 1)$.

Activity 25

The answers to parts (a), (b) and (c) can be confirmed with your calculator. The answer to the final part is that the two kinds of graph are related as mirror images in the y-axis.

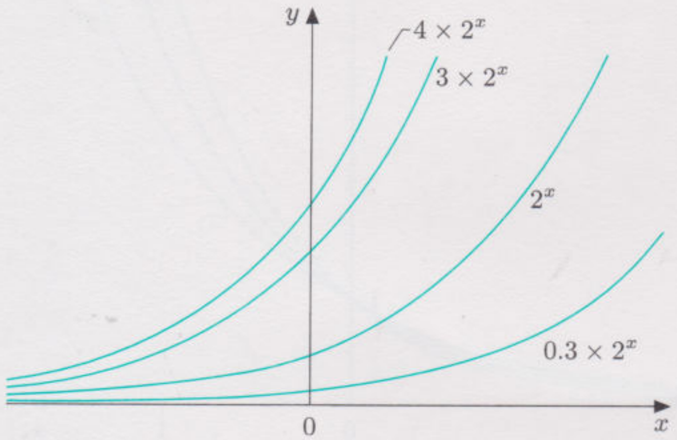
Activity 26

(a) It is the same as the graph of $y = 1$, that is, a straight line parallel to the x-axis through the point whose coordinates are $(0, 1)$.

(b) The point whose coordinates are $(0, 1)$.

Activity 27

The graphs look like the following.

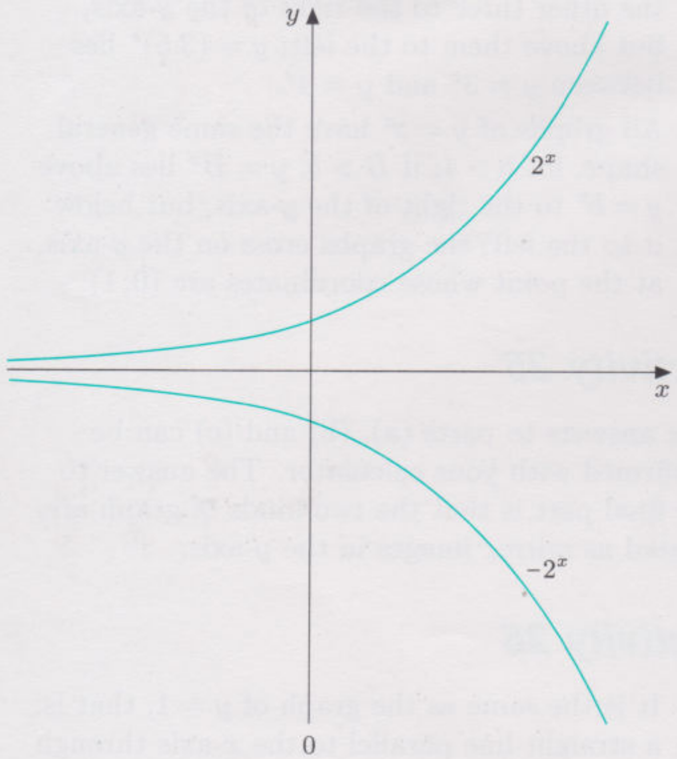


(a) 1; 3; 4; 0.3, respectively.

(b) The graphs are all similar in shape, both to each other and to the graph of $y = b^x$ for $b > 1$; but now (in contrast to a case like $y = 3^x$ and $y = 4^x$), the graph of $y = 3 \times 2^x$ lies below the graph of $y = 4 \times 2^x$ for all values of x—the two graphs never cross.

Activity 28

The graph of $y = -2^x$ looks like the following.



It is obtained by reflecting the graph of $y = 2^x$ in the x-axis.

Activity 29

(a) The whole thing is shifted down by one unit.

(b) The graph is obtained by taking that of $y = (1.05)^x$, reflecting it in the x-axis, then shifting the result up through 6 units. The graph crosses the x-axis where $y = 0$, so by finding the value of x at this point, you can say when the balance reduces to zero. The answer is about 36.7.

Activity 30

The graph of $y = 2 - 2 \times \left(\frac{1}{2}\right)^x$ climbs up to the line $y = 2$, and soon becomes indistinguishable from it.

Activity 31

(a) You simply add together the numbers of zeros in each of the factors.

(b) Ten billion millions $= 10 \times 10^9 \times 10^6 = 10^{16}$
one hundred billion billions $= 10^{20}$
one million billion thousand million billions $= 10^{33}$

Activity 32

(a) From the Earth to the Sun it is
$1.5 \times 10^8 \times 3.9 \times 10^4 = 5.85 \times 10^{12}$ inches.

(b) From Pluto to the Sun it is
$5.9 \times 10^9 \times 3.9 \times 10^4 = 23.01 \times 10^{13}$ inches; but to write this properly in scientific notation make the first factor a number between 1 and 10, so the answer is better expressed as 2.301×10^{14} inches.

Activity 33

Subtract the number of zeros in the divisor from the number of zeros in the number being divided.

Activity 34

There is no comment for this activity.

Activity 35

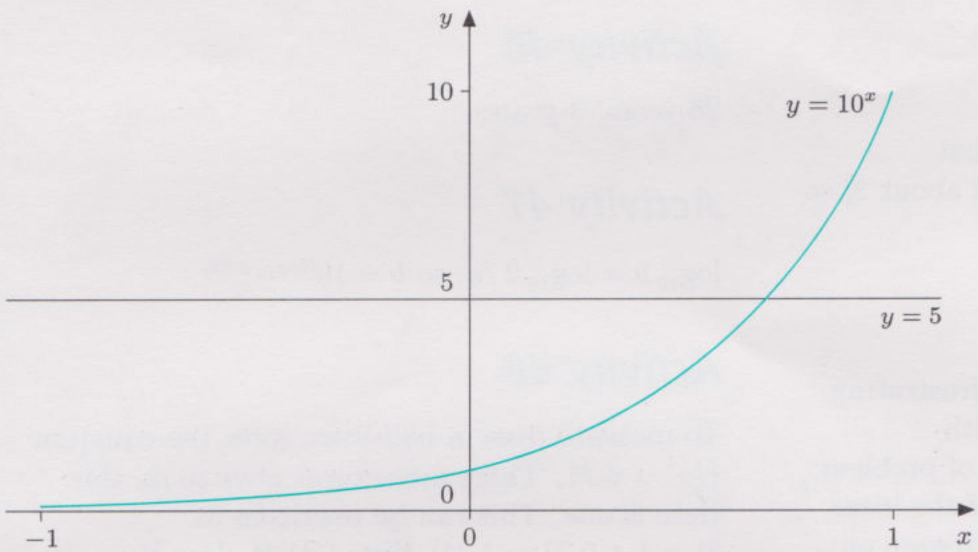
The multiplier, say s, is $2^{1/12}$. Coming down the scale, the factor is $2^{-1/12}$. There are nine semitones from C to A, so the factor relating the frequency of C to that of A is

$$(2^{-1/12})^9 = 2^{-9/12} = 2^{-3/4} = 1/(\sqrt[4]{2})^3$$

The concert pitch frequency of middle C is 262 (to the nearest whole number of cycles per second), as in Frame 2 of Section 3 of *Unit 9*.

Activity 36

The graphs of $y = 10^x$ and $y = 5$ look like the following.



The x-coordinate of the point where they intersect (which is the solution of the equation $10^x = 5$) is 0.6990, correct to four decimal places. This is found using the trace and zoom facilities in the usual way.

Activity 37

$10^{0.7} = 5.011\,872\,34$;
$10^{0.699} = 5.000\,345\,35$;
$10^{0.698\,97} = 4.999\,999\,95$;
$10^{0.698\,970\,004} = 4.999\,999\,96$

Activity 38

Since $0.7 = \frac{7}{10}$, $10^{0.7} = (\sqrt[10]{10})^7$; and since $0.699 = \frac{699}{1000}$, $10^{0.699} = (\sqrt[1000]{10})^{699}$

Activity 39

(a) $$\begin{aligned}(10^{0.699})^8 &= 10^{(0.699 \times 8)} = 10^{5.592}\\ &= 10^{0.592+5} = 10^{0.592} \times 10^5\\ &= 3.908\,089\,58 \times 10^5\end{aligned}$$

Thus, the number of letters which go out at the 8th stage is 4×10^5, approximately.

(b) The population of London is about 7×10^6. So what factor should you multiply 0.699 by to get a result which is more than 6, preferably fairly close to 7? 10 is certainly big enough, but $9 \times 0.699 = 6.291$, so 9 is probably too small. The best guess is that the number of stages required is 10.

Activity 40

$$\begin{aligned}25 = 5^2 &\simeq (10^{0.699})^2 = 10^{1.398}\\ &= 10^{0.398} \times 10^1 = 2.500\,345\,36 \times 10\end{aligned}$$

$$\begin{aligned}125 = 5^3 &\simeq (10^{0.699})^3 = 10^{2.097}\\ &= 10^{0.097} \times 10^2 = 1.250\,259\,03 \times 100\end{aligned}$$

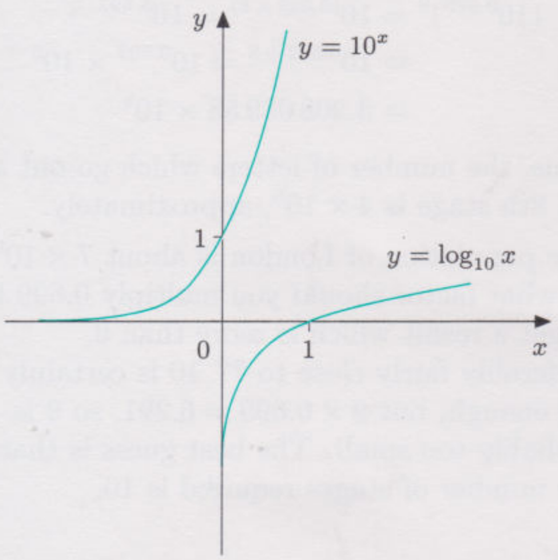
These results are not exact because $10^{0.699}$ is not exactly 5.

Activity 41

There is no comment to this activity.

Activity 42

The graphs look like this.



Activity 43

The task is to calculate 2^{30}, given that $\log 2 = 0.3$. This means that $10^{0.3}$ is about 2, so 2^{30} is about $(10^{0.3})^{30} = 10^9$.

Activity 44

Studying mathematics can be very frustrating. Some new ideas are encountered, with techniques for solving certain types of problem; there is barely time to get a sense of the ideas before it is time to practise the techniques on some exercises, and then do the relevant assignment questions. And although the techniques can be recalled a few weeks later, there is sometimes no sense of understanding. This scene is not uncommon when learning something new. It may be difficult to accept that it is often not possible to learn, understand and master an idea when you first encounter it. But stopping, taking time to think about what you have done can help to connect and crystallize ideas as well as giving you confidence to continue. In this way, you could think of learning mathematics not as a linear subject that piles up on itself (some people may feel that if they have not mastered today's work, they will not be able to understand or do tomorrow's) but more as a layering process of things becoming clearer with time and practice.

As you select examples of work, ask yourself what it shows. Does it demonstrate that you can do something or know something? Try to be as objective as you can be; if you feel that one piece of work is not relevant, select a more appropriate example.

In reviewing what you have done, note down any work or activities that you found particularly helpful.

Activity 45

14.2; 3.8. (Note that doubling the interest rate *does not* double the doubling time!)

Activity 46

28 years; 3 years.

Activity 47

$\log_{10} b = \log_{10} 2 / d$, so $b = 10^{(\log_{10} 2 / d)}$

Activity 48

To measure time in half-lives, solve the equation $\left(\frac{1}{2}\right)^t = 0.71$. There are several ways to do this. Here is one. This can be rewritten as $2^t = 1 \div 0.71 = 1.41$. Now 1.41 is close to $\sqrt{2} = 2^{1/2}$, so the required value of t is about $\frac{1}{2}$. The age of the sample is therefore about half a half-life, or roughly 3000 years.

Activity 49

That the ratio of radiocarbon to ordinary carbon in the atmosphere was the same when the carbon was fixed in the sample as it is now (an assumption which turns out not to be accurate in practice; the method has to be modified to take account of this).

Activity 50

The APR is 12.68. The monthly interest rate is 0.95% (to two decimal places).

Activity 51

The APR is $(1.01)^{24} \times 100 - 100$, which is 26.97 (to two decimal places). Note that this is a little more than the APR for a monthly interest rate of 2%.

Activity 52

The best thing to do is compute the APRs, and choose the scheme with the smallest one.

(a) The APR is 6.75.

(b) The APR is $(1.0325)^2 \times 100 - 100 = 6.61$ (to two decimal places).

(c) The APR is $(1.005)^{12} \times 100 - 100 = 6.17$ (to two decimal places).

The best option, in the absence of any other information, is the third, since it has the smallest APR. However, this conclusion might be affected by the way the loan is to be repaid, any additional costs such as service charges, and other factors not mentioned in the question. The APR can be only a guide to decisions of this nature.

Activity 53

Again, calculate the APR, but this time choose the option with the higher APR.

(a) The APR is $(1.0003)^{364} \times 100 - 100 = 11.54$.

(b) The APR is $(1.002)^{52} \times 100 - 100 = 10.95$.

The daily rate is best.

Activity 54

The completed table is as follows.

Number of times per year interest is compounded	Total sum owing at end of one year in £
1	2
2	2.25
4	2.44
12	2.61
24	2.66
52	2.69
365	2.71
500	2.7156
1000	2.7169
10000	2.7181

Activity 55

Taking ten years as the unit of time, the years become x-values as follows:

1850	0	1930	8	1990	14
1900	5	1950	10		
1925	7.5	1960	11		

The best fit is $y = 9.34 \times 10^8 \times (1.115)^x$.

Activity 56

The obvious factor is advances in medicine, which reduce infant mortality, and decrease death rates as people live longer.

Activity 57

About 64 years.

Activity 58

$\sqrt[3]{1.753} = 1.206$; 37 years.

Activity 59

There are no comments on this activity.

Activity 60

There are no comments on this activity.

Acknowledgements

Text

p. 59: Radford, T., 'Crack detective', *Guardian*, 'On Line', 2.3.1996.

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